ON THE SPECTRUM OF THE LAPLACE-BELTRAMI OPERATOR FOR p-FORMS FOR A CLASS OF WARPED PRODUCT METRICS

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1. Introduction

The spectrum of the Laplace-Beltrami operator on complete non-compact Riemannian manifolds in its connections with the geometrical properties has been investigated by many authors; however, they have mainly studied the case of scalar functions, whilst less is known about p-forms, since the complicated local expression of the Laplace-Beltrami operator on p-forms makes difficult any explicit computation. Hence, the attention has mainly focused on particular classes of Riemannian manifolds in which these difficulties can be bypassed thanks to the presence of symmetries or "asymptotic symmetries". This is the case for the hyperbolic space $\mathbb{H}^n$ ([4]), or, more generally, for rotationally symmetric Riemannian metrics ([5]) or manifolds with cylinderlike ends ([6],[7]), where a decomposition technique introduced by Dodziuk ([3]) considerably simplifies the problem. By this technique Donnelly ([4]) computed the spectrum of the Laplace-Beltrami operator on the hyperbolic space $\mathbb{H}^n$, and Eichhorn ([5], [6], [7]) obtained results on the spectrum of the Laplace-Beltrami operator for p-forms on Riemannian manifolds with cylindrical ends $M = (0, +\infty) \times N$ endowed with the Riemannian metric $d\sigma^2 = dt^2 + g(t)d\theta^2$, where $t \in (0, +\infty)$, $g(t) > 0$ and N is a compact manifold endowed with the Riemannian metric $d\theta^2$.

In the present paper, we compute the essential spectrum of the Laplace-Beltrami operator for p-forms for a class of warped product metrics. Namely, let $\overline{M}$ be a compact Riemannian n-dimensional manifold with boundary, and let y be a boundary-defining function; we endow the interior M of $\overline{M}$ with a Riemannian metric ds^2 such that in a small tubular neighbourhood of ∂M in M, ds^2 takes the form

$$ds^2 = e^{-2(a+1)t}\, dt^2 + e^{-2bt} d\theta^2_{\partial M}, \tag{1.1}$$

where $t := -\log y \in (c, +\infty)$ and $d\theta^2_{\partial M}$ is the Riemannian metric on ∂M. For $a \leq -1$ (see [8]) the Riemannian metric (1.1) is complete. Hence, the Laplace-Beltrami operator is essentially selfadjoint on the smooth, compactly supported p-forms; we compute the essential spectrum of its closure Δ^p_M in dependence on the parameters a and b.

Since Eichhorn ([5]) showed that, for any $p \in [0, n]$, the essential spectrum of Δ^p_M coincides with the essential spectrum of the Friedrichs extension $(\Delta^p_M)^F$ of Δ^p_M on smooth p-forms with compact support in $(c, +\infty) \times \partial M$, we compute the essential spectrum of $(\Delta^p_M)^F$. To this purpose we use an orthogonal decomposition of $L^2_p((c, +\infty) \times \partial M)$ analogue to those employed by Eichhorn and Donnelly. The decomposition is carried out in two steps: first, thanks to the Hodge decomposition

on ∂M, we write any p-form ω on $(c,+\infty)\times\partial M$ as

$$\omega=\omega_{1\delta}\oplus\omega_{2d}\wedge dt\oplus(\omega_{1d}\oplus\omega_{2\delta}\wedge dt), \tag{1.2}$$

where $\omega_{1\delta}$ (resp. ω_{1d}) is a coclosed (resp. closed) p-form on ∂M parametrized by t and $\omega_{2\delta}$ (resp. ω_{2d}) is a coclosed (resp. closed) $(p-1)$-form on ∂M parametrized by t. This yields the orthogonal decomposition

$$\begin{aligned}(1.3)\quad & L^2_p((c,+\infty)\times\partial M)=\\ & \mathcal{L}_1((c,+\infty)\times\partial M)\oplus\mathcal{L}_2((c,+\infty)\times\partial M)\oplus\mathcal{L}_3((c,+\infty)\times\partial M),\end{aligned}$$

where for any $\omega\in L^2_p((c,+\infty)\times\partial M)$

$$\omega_{1\delta}\in\mathcal{L}_1((c,+\infty)\times\partial M),$$

$$\omega_{2d}\wedge dt\in\mathcal{L}_2((c,+\infty)\times\partial M),$$

and

$$(\omega_{1d}\oplus\omega_{2\delta}\wedge dt)\in\mathcal{L}_3((c,+\infty)\times\partial M).$$

Since Δ^p_M is invariant under (1.3), we get the corresponding decomposition of $(\Delta^p_M)^F$

$$(\Delta^p_M)^F=(\Delta^p_{M1})^F\oplus(\Delta^p_{M2})^F\oplus(\Delta^p_{M3})^F,$$

whence

$$\sigma_{\mathrm{ess}}(\Delta^p_M)=\sigma_{\mathrm{ess}}((\Delta^p_M)^F)=\bigcup_{i=1}^{3}\sigma_{\mathrm{ess}}((\Delta^p_{Mi})^F).$$

Hence, the calculus of the essential spectrum of Δ^p_M can be reduced to the determination of the essential spectra of $(\Delta^p_{Mi})^F$, for $i=1,2,3$. Moreover, since the Hodge $*$ operator maps isometrically p-forms of $\mathcal{L}_1((c,+\infty)\times\partial M)$ onto $(n-p)$-forms of $\mathcal{L}_2((c,+\infty)\times\partial M)$, it suffices to consider the cases $i=1,3$.

The second step consists in the decomposition of $\omega_{1\delta}$ (resp. of ω_{2d}, $\omega_{2\delta}$) according to an orthonormal basis of coclosed p-eigenforms (resp. closed and coclosed $(p-1)$-eigenforms) of $\Delta^p_{\partial M}$ (resp. $\Delta^{p-1}_{\partial M}$) on ∂M. In this way, up to a unitary equivalence, the spectral analysis of $(\Delta^p_{Mi})^F$, $i=1,3$, can be reduced to the investigation of the spectra of a countable number of Sturm-Liouville operators $(D_{i\lambda})^F$ on $(c,+\infty)$, parametrized by the eigenvalues λ^p_k of $\Delta^p_{\partial M}$ if $i=1$, and by the eigenvalues λ^{p-1}_k of $\Delta^{p-1}_{\partial M}$ if $i=3$. We remark that, since we deal with an infinite family of operators, we only get the inclusion

$$\bigcup_{k=1}^{\infty}\sigma_{\mathrm{ess}}((D_{i\lambda_k})^F)\subseteq\sigma_{\mathrm{ess}}((\Delta^p_{Mi})^F).$$

In order to fully determine the essential spectrum, we need additional estimates on the behaviour of the isolated eigenvalues of finite multiplicity of $(D_{i\lambda_k})^F$ as $k \to +\infty$.

For $i = 1$ the situation is simpler; the essential spectrum of $(D_{1\lambda})^F$ can be easily computed, and when necessary we can easily deduce the estimates on the isolated eigenvalues of finite multiplicity. Moreover, as already noted by Eichhorn ([6]), this part of the spectrum depends strongly on the geometry of M (more precisely, on the asymptotic sectional curvature).

On the other hand, the analysis of $(\Delta^p_{M3})^F$ is difficult, since the operators $(D_{3\lambda})^F$ are coupled systems of Sturm-Liouville operators on $L^2(c, +\infty) \oplus L^2(c, +\infty)$. However, the essential spectrum of $D^F_{3\lambda}$ can be determined through perturbation theory, taking, when necessary, as "unperturbed operator" an operator D^F_{30} which is invariant under the transformation $(w_1 \oplus w_2) \longrightarrow (w_2 \oplus w_1)$. Indeed, due to this invariance, the essential spectrum of D^F_{30} can be computed as the union of the essential spectra of its restrictions to the subspaces $\mathcal{V}_1 = \{w_1 \oplus w_2 \mid w_1 = w_2\}$ and $\mathcal{V}_2 = \{w_1 \oplus w_2 \,|\, w_1 = -w_2\}$ of $L^2(c, +\infty) \oplus L^2(c, +\infty)$.

More difficult is to get the estimates on the behaviour of the isolated eigenvalues of finite multiplicity of $(D_{3\lambda_k})^F$ as $k \to +\infty$. However, this difficulty can be overcome observing that, roughly speaking, the differential $d^p_M\omega$ of a smooth p-eigenform ω of Δ^p_M in $\mathcal{L}_3((c, +\infty) \times \partial M)$ is a smooth $(p+1)$-eigenform of Δ^{p+1}_M in $\mathcal{L}_2((c, +\infty) \times \partial M)$, whilst its codifferential $\delta^p_M\omega$ is a smooth $(p-1)$-eigenform of Δ^{p-1}_M in $\mathcal{L}_1((c, +\infty) \times \partial M)$. This entails the existence of a link between the positive parts of the spectra for forms of different types and degrees. Namely, one can prove that if there exists a sequence $\{\mu_k\}$ of positive eigenvalues of $(\Delta^p_{M3})^F$ and a corresponding sequence of eigenforms $\{\Phi_k\}$ such that, for every $k \in \mathbb{N}$, $(\Delta^p_{M3})^F\Phi_k - \mu_k\Phi_k = 0$ and $\mu_k \to \mu \geq 0$ as $k \to +\infty$, then either μ is in the essential spectrum of $(\Delta^{(p-1)}_{M1})^F$ or μ is in the essential spectrum of $(\Delta^{(p+1)}_{M2})^F$. This observation, combined with the information which we already have on the essential spectra of $(\Delta^p_{M1})^F$ and $(\Delta^p_{M2})^F$ for every degree p, allows to compute completely $\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\}$.

However, in this way it is not possible to determine if 0 lies or not in the essential spectrum; indeed, we cannot decide whether 0 is an eigenvalue of infinite multiplicity of $(\Delta^p_{M3})^F$. This can be established, instead, if there is a rotational symmetry, that is, if $\overline{M}$ is the unitary ball $\overline{B(\underline{0}, 1)}$ in $\mathbb{R}^n$ and the Riemannian metric is globally invariant under rotations. In fact, if the Riemannian metric coincides with the

Euclidean metric in a small neighbourhood $B(\underline{0}, \epsilon)$ of $\underline{0}$, a slight modification of a classical result of Dodziuk ([3]) allows to check whether $0 \in \sigma_p(\Delta^p_M)$ and, if so, to determine its multiplicity. Adding this new information, in this case we can describe completely the essential spectrum of Δ^p_M.

Now we discuss briefly our results. If $a = -1$, the infimum of $\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\}$ for a given dimension n depends on the degree p of the forms. Since there is a strong dependence of the spectral properties of the operators on the sign of b, we consider separately the cases $b < 0$, $b = 0$, $b > 0$.

If $a = -1$ and $b < 0$, the situation is similar to the hyperbolic case (see [4], [1]). With no rotational symmetry assumptions,

$$\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\} =$$

$$= \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right),$$

whilst in the rotationally symmetric case

$$\sigma_{\text{ess}}(\Delta^p_M) = \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right)$$

for $p \neq \frac{n}{2}$ and

$$\sigma_{\text{ess}}(\Delta^p_M) = \{0\} \cup \left[\frac{b^2}{4}, +\infty\right)$$

for $p = \frac{n}{2}$.

If $a = -1$ and $b = 0$, we get the well-known result for cylindrical ends

$$\sigma_{\text{ess}}(\Delta^p_M) = \bigcup_k \left([\lambda^p_k, +\infty) \cup [\lambda^{p-1}_k, +\infty)\right),$$

where λ^p_k (resp. λ^{p-1}_k) denote as usual the eigenvalues of the Laplace-Beltrami operator $\Delta^p_{\partial M}$ (resp. $\Delta^{p-1}_{\partial M}$) on ∂M.

If $a = -1$ and $b > 0$, the essential spectrum of Δ^p_M somewhat reflects the cohomology of ∂M. Indeed, if both the p-th and the $(p-1)$-th Betti numbers of ∂M vanish, $\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\} = \emptyset$; if the p-th Betti number of ∂M is different from zero while the $(p-1)$-th Betti number of ∂M vanishes,

$$\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\} = \left[\left(\frac{n-2p-1}{2}\right)^2 b^2, +\infty\right);$$

if the p-th Betti number of ∂M vanishes while the $(p-1)$-th Betti number of ∂M is different from zero,

$$\sigma_{\mathrm{ess}}(\Delta_M^p) \setminus \{0\} = \left[\left(\frac{n-2p+1}{2} \right)^2 b^2, +\infty \right);$$

finally, if both the p-th and the $(p-1)$-th Betti numbers of ∂M are different from zero,

$$\sigma_{\mathrm{ess}}(\Delta_M^p) \setminus \{0\} =$$

$$= \left[\min \left\{ \left(\frac{n-2p-1}{2} \right)^2 b^2, \left(\frac{n-2p+1}{2} \right)^2 b^2 \right\}, +\infty \right).$$

In particular, in the rotationally symmetric case, where $\partial M = \mathbb{S}^{n-1}$, we find that $0 \notin \sigma_{\mathrm{ess}}(\Delta_M^p)$; hence, for $1 < p < n-1$ $\sigma_{\mathrm{ess}}(\Delta_M^p) = \emptyset$, whilst for $p \in \{0, 1, n-1, n\}$ $\sigma_{\mathrm{ess}}(\Delta_M^p) = \left[\left(\frac{n-1}{2} \right)^2 b^2, +\infty \right)$.

If $a < -1$, for $b \neq 0$ the infimum of $\sigma_{\mathrm{ess}}(\Delta_M^p) \setminus \{0\}$ is 0. Again, much depends on the sign of b.

If $b < 0$, for every $p \in [0, n]$, $\sigma_{\mathrm{ess}}(\Delta_M^p) = [0, +\infty)$.

If $b = 0$, as before,

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = \bigcup_k \left([\lambda_k^p, +\infty) \cup [\lambda_k^{p-1}, +\infty) \right).$$

If finally $b > 0$, also for $a < -1$ the essential spectrum of Δ_M^p reflects somehow the cohomology of the boundary ∂M. As a matter of fact, if both the p-th and the $(p-1)$-th Betti numbers of ∂M vanish, $\sigma_{\mathrm{ess}}(\Delta_M^p) \setminus \{0\} = \emptyset$. If, on the contrary, at least one among the p-th and the $(p-1)$-th Betti numbers of ∂M is different from zero, then $\sigma_{\mathrm{ess}}(\Delta_M^p) = [0, +\infty)$. As a consequence, in the rotationally symmetric case, where $\partial M = \mathbb{S}^{n-1}$, if $1 < p < n-1$ $\sigma_{\mathrm{ess}}(\Delta_M^p) = \emptyset$, whilst if $p \in \{0, 1, n-1, n\}$ $\sigma_{\mathrm{ess}}(\Delta_M^p) = [0, +\infty)$.

A more detailed analysis might yield a description of the absolutely continuous spectrum of Δ_M^p. However, at least for the time being, this turns out to be very difficult because the Fourier transform, which is a basic tool in scattering theory, is not available in this context.

The paper is organized as follows: in section 2 we introduce some preliminary facts and basic notations. In section 3 we describe in some details the decomposition techniques. The actual calculus of the essential spectrum (with 0 excluded) in the general case is developed in section 4 for $a = -1$ and in section 5 for $a < -1$. In section 6, finally, we fully determine the essential spectrum of Δ_M^p in the rotationally symmetric case.

2. Preliminary facts and notations

Let $\overline{M}$ be a compact, n-dimensional manifold with boundary, and let y be a positive defining function for $N := \partial M$:

$$y \geq 0, \qquad y^{-1}(0) = N, \qquad dy_{|N} = 0.$$

We endow the interior M of $\overline{M}$ with a Riemannian metric which, in a tubular neighbourhood $(0, \varepsilon) \times N$ of N in M, is given by

$$d\sigma^2 = y^{2a}\, dy^2 + y^{2b}\, d\theta_N^2, \tag{2.1}$$

where $d\theta_N^2$ is a Riemannian metric on N.

Under the change of variables $t = -\log y$, (2.1) is trasformed into the Riemannian metric

$$ds^2 = e^{-2(a+1)t}\, dt^2 + e^{-2bt}\, d\theta_N^2, \tag{2.2}$$

defined on $(c, +\infty) \times N$, where $c = -\log \varepsilon$.

It is well-known (see [8]) that a Riemannian metric of this kind is complete if and only if $a \leq -1$. Hence, throughout the paper, we will suppose that $a \leq -1$.
For $p = 0, ..., n$, we will denote by $C^\infty(\Lambda^p(M))$ (resp. $C^\infty(\Lambda^p\,((c, +\infty) \times N)))$ the space of all the smooth p-forms on M (resp. on $(c, +\infty) \times N$), and by $C_c^\infty(\Lambda^p(M))$ (resp. $C_c^\infty(\Lambda^p((c, +\infty) \times N)))$ the set of all the smooth p-forms with compact support in M (resp. in $(c, +\infty) \times N$).

For any $\omega \in C^\infty(\Lambda^p(M))$, we will denote by $|\omega(x)|$ the norm induced by the Riemannian metric on the fiber over x, given in local coordinates by

$$|\omega(x)|^2 = g^{i_1 j_1}(x) ... g^{i_p j_p}(x) \omega_{i_1 ... i_p}(x) \omega_{j_1 ... j_p}(x),$$

where g^{ij} is the expression of the Riemannian metric in local coordinates. Following [2], we will denote by d_M^p, $*_M$, δ_M^p respectively the differential, the Hodge $*$ operator and the codifferential for p-forms on M. Δ_M^p will stand for the Laplace-Beltrami operator acting on p-forms

$$\Delta_M^p = d_M^{p-1} \delta_M^p + \delta_M^{p+1} d_M^p,$$

which is expressed in local coordinates by the Weitzenböck formula

$$(\Delta_M^p \omega)_{i_1 ... i_p} = -g^{ij} \nabla_i \nabla_j \omega_{i_1 ... i_p} + \sum_j R_j^\alpha \omega_{i_1 ... \alpha ... i_p} + \sum_{j, l \neq j} R^{\alpha\,\beta}_{i_j\,i_l} \omega_{\alpha i_1 ... \beta ... i_p},$$

where $\nabla_i \omega$ is the covariant derivative of ω with respect to the Riemannian metric, and R_j^i, $R^{i\,j}_{k\,l}$ denote respectively the local components of the Ricci tensor and of the Riemann tensor induced by the Riemannian metric.

As usual, $L^2_p(M)$ will denote the completion of $C_c^\infty(\Lambda^p(M))$ with respect to the norm $\|\omega\|_{L^2_p(M)}$ induced by the scalar product

$$\langle \omega, \tilde{\omega} \rangle_{L^2_p(M)} := \int_M \omega \wedge *_M \tilde{\omega};$$

$\|\omega\|_{L^2_p(M)}$ can also be written as

$$\|\omega\|^2_{L^2_p(M)} = \int_M |\omega(t,\theta)|^2 dV_M,$$

where dV_M is the volume element of (M, ds^2).

It is well-known that, since the Riemannian metric on M is complete, the Laplace-Beltrami operator is essentially selfadjoint on $C_c^\infty(\Lambda^p(M))$, for $p = 0, ..., n$. We will denote by the same symbol Δ^p_M its closure, and we will study its essential spectrum.

We recall that, given a selfadjoint operator $A : \mathcal{D}(A) \to \mathcal{H}$ in a Hilbert space $\mathcal{H}$, its essential spectrum is defined as the set

$$\sigma_{\text{ess}}(A) := \sigma(A) \setminus \sigma_d(A),$$

where $\sigma_d(A)$, called the discrete spectrum of A, is the set of isolated eigenvalues of finite multiplicity of A. The essential spectrum of A can be also characterized in terms of sequences as follows: $\mu \in \sigma_{\text{ess}}(A)$ if and only if there exists a Weyl sequence for μ, that is, a sequence $\{w_n\} \subset \mathcal{D}(A)$ with no convergent subsequences in $\mathcal{H}$, bounded in $\mathcal{H}$ and such that

$$\lim_{n\to+\infty} (A - \mu) w_n = 0 \quad \text{in } \mathcal{H}.$$

If A is a selfadjoint operator on a Hilbert space $\mathcal{H}$, an operator C such that $\mathcal{D}(A) \subset \mathcal{D}(C)$ is called relatively compact with respect to A if C is compact from $\mathcal{D}(A)$ with the graph norm $\|.\|_A$ given by

$$\|\phi\|^2_A = \|\phi\|^2_{\mathcal{H}} + \|A\phi\|^2_{\mathcal{H}}$$

to $\mathcal{H}$ with the norm $\|.\|_{\mathcal{H}}$. Moreover (see [10]), given a selfadjoint operator A on a Hilbert space $\mathcal{H}$, and a symmetric operator C relatively compact with respect to A, if $B = A + C$ is selfadjoint on $\mathcal{D}(A)$, then

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B).$$

For any symmetric, positive operator $A : \mathcal{D}(A) \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$, we will denote by A^F its Friedrichs extension, defined by

$$\mathcal{D}(A^F) = X_A \cap \mathcal{D}(A^*),$$

$$A^F w = A^* w,$$

where X_A denotes the completion of $\mathcal{D}(A)$ with respect to the graph norm.

In particular, we will denote by $(\Delta^p_{M,c})^F$ the Friedrichs extension of the restriction of Δ^p_M to $C^\infty_c(\Lambda^p((c,+\infty)\times N))$.

In [5], J. Eichhorn showed that the essential spectrum of the Laplace-Beltrami operator acting on p-forms on a complete noncompact Riemannian manifold is equal to the essential spectrum of the Friedrichs extension of the restriction of the same operator to the smooth p-forms with compact support in $M\setminus K$, where $K\subset M$ is any compact subset of M. In the present case, this yields

$$\sigma_{\mathrm{ess}}(\Delta^p_M)=\sigma_{\mathrm{ess}}((\Delta^p_{M,c})^F),$$

where $c>-\log\varepsilon$ can be taken arbitrarily large.

3. Hodge decomposition

In the present section let us suppose, more generally, that the Riemannian metric ds^2 in $(c,+\infty)\times N$, takes the form

$$ds^2=f(t)\,dt^2+g(t)\,d\theta^2_N, \tag{3.1}$$

where $f(t)>0$ and $g(t)>0$ for any $t\in(c,+\infty)$. Moreover, we will suppose that $c>0$ is fixed and we will write $(\Delta^p_M)^F$ for $(\Delta^p_{M,c})^F$.

Now, any $\omega\in C^\infty_c(\Lambda^p((c,+\infty)\times N))$ can be decomposed as

$$\omega=\omega_1+\omega_2\wedge dt, \tag{3.2}$$

where ω_1 and ω_2 are respectively a p-form and a $(p-1)$-form on N depending on t. An easy computation shows that $*_M\omega$ can be expressed in terms of decomposition (3.2) as

$$*_M\omega=(-1)^{n-p}g^{\frac{n-2p+1}{2}}(t)f^{-\frac12}(t)*_N\omega_2+g^{\frac{n-2p-1}{2}}(t)f^{\frac12}(t)*_N\omega_1\wedge dt, \tag{3.3}$$

where $*_N$ denotes the Hodge $*$ operator on N. Moreover, d^p_M and δ^p_M split respectively as

$$d^p_M\omega=d^p_N\omega_1+\left\{(-1)^p\frac{\partial\omega_1}{\partial t}+d^{p-1}_N\omega_2\right\}\wedge dt, \tag{3.4}$$

$$\delta^p_M\omega=g^{-1}(t)\delta^p_N\omega_1+(-1)^pf^{-\frac12}g^{\frac{-n-1+2p}{2}}\frac{\partial}{\partial t}\left(f^{-\frac12}g^{\frac{n+1-2p}{2}}\omega_2\right)+ \\ +g^{-1}\delta^{p-1}_N\omega_2\wedge dt, \tag{3.5}$$

where d^p_N is the differential on N and δ^p_N is the codifferential on N.

Moreover, the L^2-norm of $\omega \in C^\infty(\Lambda^p((c,+\infty)\times N))\cap L^2_p((c,+\infty)\times N)$ can be written as

$$\|\omega\|^2_{L^2_p((c,+\infty)\times N)} = \int_c^{+\infty} g^{\frac{n-2p-1}{2}}(s) f^{\frac{1}{2}}(s)\|\omega_1(s)\|^2_{L^2_p(N)}\,ds + \\ + \int_c^{+\infty} g^{\frac{n+1-2p}{2}}(s) f^{-\frac{1}{2}}(s)\|\omega_2(s)\|^2_{L^2_{p-1}(N)}\,ds, \tag{3.6}$$

where $\|.\|_{L^2_p(N)}$ is the L^2-norm for p-forms on N.

From (3.4) and (3.5), a lengthy but straightforward computation gives

$$\Delta^p_M\omega = (\Delta^p_M\omega)_1 + (\Delta^p_M\omega)_2\wedge dt,$$

with

$$(\Delta^p_M\omega)_1 = g^{-1}(t)\Delta^p_N\omega_1 + (-1)^p f^{-1}(t)g^{-1}(t)\frac{\partial g}{\partial t}d^{p-1}_N\omega_2 + \\ - f^{-\frac{1}{2}}(t)g^{\frac{-n+1+2p}{2}}(t)\frac{\partial}{\partial t}\left(f^{-\frac{1}{2}}(t)g^{\frac{n-1-2p}{2}}(t)\frac{\partial\omega_1}{\partial t}\right) \tag{3.7}$$

and

$$(\Delta^p_M\omega)_2 = g^{-1}(t)\Delta^{p-1}_N\omega_2 + (-1)^p g^{-2}(t)\frac{\partial g}{\partial t}\delta^p_N\omega_1 + \\ - \frac{\partial}{\partial t}\left\{f^{-\frac{1}{2}}(t)g^{\frac{-n-1+2p}{2}}(t)\frac{\partial}{\partial t}\left(f^{-\frac{1}{2}}(t)g^{\frac{n+1-2p}{2}}(t)\omega_2\right)\right\}, \tag{3.8}$$

where Δ^p_N (resp. Δ^{p-1}_N) stands for the Laplace-Beltrami operator for p-forms (resp. for $(p-1)$-forms) on N.

Since for every smooth $\omega \in L^2_p((c,+\infty)\times N)$ we have that $\omega_1 \in L^2_p((c,+\infty)\times N)$, $\omega_2\wedge dt \in L^2_p((c,+\infty)\times N)$ and

$$\langle \omega_1, \omega_2\wedge dt\rangle_{L^2_p((c,+\infty)\times N)} = 0,$$

(3.2) gives rise to an orthogonal decomposition of $L^2_p((c,+\infty)\times N)$ into two closed subspaces. However, (3.7) and (3.8) show that Δ^p_M is not invariant under this decomposition. As a consequence, further decompositions are required.

It is well-known that, for $0\le p\le n-1$,

$$C^\infty(\Lambda^p(N)) = dC^\infty(\Lambda^{p-1}(N)) \oplus \delta C^\infty(\Lambda^{p+1}(N)) \oplus \mathcal{H}^p(N),$$

where $\mathcal{H}^p(N)$ is the space of harmonic p-forms on N, and the decomposition is orthogonal in $L^2_p(N)$. Hence, for $0\le p\le n-1$,

$$L^2_p(N) = \overline{dC^\infty(\Lambda^{p-1}(N))} \oplus \overline{\delta C^\infty(\Lambda^{p+1}(N))} \oplus \mathcal{H}^p(N).$$

As a consequence, for $1 \leq p \leq n-1$, every $\omega \in L^2_p((c,+\infty) \times N)$ can be written as

$$\omega = \omega_{1\delta} \oplus \omega_{2d} \wedge dt \oplus (\omega_{1d} \oplus \omega_{2\delta} \wedge dt),$$

where $\omega_{1\delta}$ (resp. ω_{1d}) is a coclosed (resp. closed) p-form on N parametrized by t, and $\omega_{2\delta}$ (resp. ω_{2d}) is a coclosed (resp. closed) $(p-1)$-form on N parametrized by t. In this way we get the orthogonal decomposition

$$L^2_p((c,+\infty) \times N) =$$
$$= \mathcal{L}_1((c,+\infty) \times N) \oplus \mathcal{L}_2((c,+\infty) \times N) \oplus \mathcal{L}_3((c,+\infty) \times N),$$

where, for every $\omega \in L^2_p((c,+\infty) \times N)$,

$$\omega_{1\delta} \in \mathcal{L}_1((c,+\infty) \times N),$$

$$\omega_{2d} \wedge dt \in \mathcal{L}_2((c,+\infty) \times N)$$

and

$$(\omega_{1d} \oplus \omega_{2\delta} \wedge dt) \in \mathcal{L}_3((c,+\infty) \times N).$$

Since

$$d^p_N \Delta^p_N = \Delta^{p+1}_N d^p_N, \qquad \delta^p_N \Delta^p_N = \Delta^{p-1}_N \delta^p_N,$$
$$\frac{\partial}{\partial t} d^p_N = d^p_N \frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial t} \delta^p_N = \delta^p_N \frac{\partial}{\partial t},$$

the Laplace-Beltrami operator is invariant under this decomposition, and

$$(\Delta^p_M)^F = (\Delta^p_{M1})^F \oplus (\Delta^p_{M2})^F \oplus (\Delta^p_{M3})^F,$$

where, for $i = 1,2,3$, $(\Delta^p_{Mi})^F$ is the Friedrichs extension of the restriction of Δ^p_M to $C^\infty_c(\Lambda^p((c,+\infty) \times N)) \cap \mathcal{L}_i((c,+\infty) \times N)$.

Since the orthogonal sum is finite, for $1 \leq p \leq n-1$,

$$\sigma_{\text{ess}}((\Delta^p_M)^F) = \bigcup_{i=1}^{3} \sigma_{\text{ess}}((\Delta^p_{Mi})^F),$$

$$\sigma_p((\Delta^p_M)^F) = \bigcup_{i=1}^{3} \sigma_p((\Delta^p_{Mi})^F).$$

For $p = 0$ (resp. $p = n$), any $\omega \in L^2_p((c,+\infty) \times N))$ can be written as $\omega = \omega_{1\delta}$ (resp. $\omega = \omega_{2d} \wedge dt$), where $\omega_{1\delta}$ (resp. ω_{2d}) is a coclosed (resp. closed) 0-form (resp. $(n-1)$-form) parametrized by t on N. Hence $L^2_0((c,+\infty) \times N) = \mathcal{L}_1((c,+\infty) \times N)$ (resp. $L^2_n((c,+\infty) \times N) = \mathcal{L}_2((c,+\infty) \times N)$), and $(\Delta^p_M)^F = (\Delta^p_{M1})^F$ (resp. $(\Delta^p_M)^F = (\Delta^p_{M2})^F$).

Hence, for any $p \in [0,n]$, in order to determine the spectrum of $(\Delta^p_M)^F$ it suffices to study the spectral properties of $(\Delta^p_{Mi})^F$, $i = 1,2,3$.

To this purpose, let us introduce further decompositions. First of all, we decompose $\omega_{1\delta}$ according to an orthonormal basis $\{\tau_{1k}\}_{k\in\mathbb{N}}$ of coclosed p-eigenforms of Δ^p_N; this yields

$$\omega_{1\delta} = \oplus_k h_k(t)\tau_{1k}, \tag{3.9}$$

where $h_k(t)\tau_{1k} \in L^2_p((c,+\infty)\times N)$ for every $k \in \mathbb{N}$. The sum (3.9) is orthogonal in $L^2_p((c,+\infty)\times N)$, thanks to (2.2). We will call p-form of type I any p-form $\omega \in L^2_p((c,+\infty)\times N)$ such that

$$\omega = h(t)\tau_1,$$

where τ_1 is a coclosed normalized p-eigenform, corresponding to some eigenvalue λ of Δ^p_N. For every $k \in \mathbb{N}$, let us denote by λ^p_k the eigenvalue associated to τ_{1k}. Since for every $k \in \mathbb{N}$

$$\begin{aligned}\Delta^p_{M1}(h(t)\tau_{1k}) = {} & \frac{\lambda^p_k}{g(t)}h(t)\tau_{1k} \\ & - f(t)^{-\frac{1}{2}}g(t)^{\frac{-n+1+2p}{2}}\frac{\partial}{\partial t}\left(f(t)^{-\frac{1}{2}}g(t)^{\frac{n-1-2p}{2}}\frac{\partial h}{\partial t}\right)\tau_{1k},\end{aligned} \tag{3.10}$$

Δ^p_{M1} is invariant under decomposition (3.9), and, since, if $\omega = h(t)\tau_{1k}$,

$$\|\omega\|^2_{L^2_p((c,+\infty)\times N)} = \int_c^\infty g(s)^{\frac{n-2p-1}{2}} f(s)^{\frac{1}{2}} h(s)^2\, ds,$$

$(\Delta^p_{M1})^F$ is unitarily equivalent to the direct sum over $k \in \mathbb{N}$ of the Friedrichs extensions $(\Delta_{1\lambda^p_k})^F$ of the operators

$$\Delta_{1\lambda^p_k} : C^\infty_c(c,+\infty) \longrightarrow L^2((c,+\infty), g^{\frac{n-2p-1}{2}} f^{\frac{1}{2}})$$

$$\Delta_{1\lambda^p_k} h = \left\{\frac{\lambda^p_k}{g(t)}h(t) - f(t)^{-\frac{1}{2}}g(t)^{\frac{-n+1+2p}{2}}\frac{\partial}{\partial t}\left(f(t)^{-\frac{1}{2}}g(t)^{\frac{n-1-2p}{2}}\right)\right\}. \tag{3.11}$$

If we introduce the tranformation

$$w(t) = h(t)f(t)^{\frac{1}{4}}g(t)^{\frac{n-2p-1}{4}}, \tag{3.12}$$

a direct (but lengthy) computation shows that $(\Delta^p_{M1})^F$ is unitarily equivalent to the direct sum, over $k \in \mathbb{N}$, of the Friedrichs extensions $(D_{1\lambda^p_k})^F$ of the operators

$$D_{1\lambda^p_k} : C^\infty_c(c,+\infty) \longrightarrow L^2(c,+\infty)$$

given by

$$(3.13)\quad D_{1\lambda_k^p} w = -\frac{\partial}{\partial t}\left(\frac{1}{f}\frac{\partial w}{\partial t}\right) + \Bigg\{-\frac{7}{16}\frac{1}{f^3}\left(\frac{\partial f}{\partial t}\right)^2 + \frac{1}{4}\frac{1}{f^2}\frac{\partial^2 f}{\partial t^2} + $$
$$-\frac{1}{2}\frac{1}{f^2}\frac{\partial f}{\partial t}\frac{(n-1-2p)}{4}\frac{1}{g}\frac{\partial g}{\partial t} + \frac{1}{f}\frac{(n-2p-1)}{4}\frac{(n-2p-5)}{4}\frac{1}{g^2}\left(\frac{\partial g}{\partial t}\right)^2 +$$
$$+\frac{1}{f}\frac{(n-2p-1)}{4}\frac{1}{g}\frac{\partial^2 g}{\partial t^2} + \frac{\lambda_k^p}{g}\Bigg\} w.$$

Analogously, we decompose ω_{2d} according to an orthonormal basis of closed $(p-1)$-eigenforms $\{\tau_{2k}\}_{k\in\mathbb{N}}$ of Δ_N^{p-1}:

$$(3.14)\qquad \omega_{2d} \wedge dt = \oplus_k h_k(t)\tau_{2k} \wedge dt.$$

We will call p-form of type II any p-form $\omega \in L^2_p((c,+\infty)\times N)$ such that

$$\omega = h(t)\tau_2 \wedge dt,$$

where τ_2 is a coclosed normalized $(p-1)$-eigenform, corresponding to some eigenvalue λ of Δ_N^{p-1}. For every $k \in \mathbb{N}$

$$\Delta^p_{M2}(h(t)\tau_{2k}\wedge dt) = (\Delta_{2\lambda_k^{p-1}} h)\tau_{2k}\wedge dt,$$

where

$$(3.15)\quad \Delta_{2\lambda_k^{p-1}} h = \frac{\lambda_k^{p-1}}{g(t)} h(t) -$$
$$\frac{\partial}{\partial t}\left\{ f(t)^{-\frac{1}{2}} g(t)^{\frac{-n-1+2p}{2}} \frac{\partial}{\partial t}\left(f(t)^{-\frac{1}{2}} g(t)^{\frac{n+1-2p}{2}} h(t)\right)\right\}.$$

Here, again, for every $k \in \mathbb{N}$ we denote by λ_k^{p-1} the eigenvalue of Δ_N^{p-1} corresponding to the eigenform τ_{2k}. Since if $\omega = h(t)\tau_{2k}\wedge dt$

$$\|\omega\|^2_{L^2_p((c,+\infty)\times N)} = \int_c^\infty g(s)^{\frac{n-2p+1}{2}} f(s)^{-\frac{1}{2}} h(s)^2\, ds,$$

introducing the transformation

$$(3.16)\qquad w(t) = h(t) f(t)^{-\frac{1}{4}} g(t)^{\frac{n+1-2p}{4}},$$

we find that $(\Delta^p_{M2})^F$ is unitarily equivalent to the direct sum, over $k \in \mathbb{N}$, of the Friedrichs extensions $(D_{2\lambda_k^{p-1}})^F$ of the operators

$$D_{2\lambda_k^{p-1}} : C_c^\infty(c,+\infty) \longrightarrow L^2(c,+\infty)$$

$$(3.17)\quad D_{2\lambda_k^{p-1}}w=-\frac{\partial}{\partial t}\left(\frac{1}{f}\frac{\partial w}{\partial t}\right)+\left\{-\frac{7}{16}\frac{1}{f^3}\left(\frac{\partial f}{\partial t}\right)^2+\frac{1}{4}\frac{1}{f^2}\frac{\partial^2 f}{\partial t^2}+\right.$$

$$-\frac{1}{2}\frac{1}{f^2}\frac{\partial f}{\partial t}\frac{(n-1+2p)}{4}\frac{1}{g}\frac{\partial g}{\partial t}+\frac{1}{f}\frac{(n-2p+1)}{4}\frac{(n-2p+5)}{4}\frac{1}{g^2}\left(\frac{\partial g}{\partial t}\right)^2+$$

$$\left.+\frac{1}{f}\frac{(-n+2p-1)}{4}\frac{1}{g}\frac{\partial^2 g}{\partial t^2}+\frac{\lambda_k^{p-1}}{g}\right\}w.$$

Decompose now $\omega_{2\delta}$ with respect to an orthonormal basis of coclosed $(p-1)$-eigenforms $\{\tau_{3k}\}_{k\in\mathbb{N}}$ of Δ_N^{p-1}; for every $k\in\mathbb{N}$, denote by λ_k^{p-1} the eigenvalue corresponding to the eigenform τ_{3k}.

Then $\left\{\frac{1}{\sqrt{\lambda_k^{p-1}}}d_N\tau_{3k}\right\}_{k\in\mathbb{N}}$ is an orthonormal basis of closed eigenforms on N, and we get the following decomposition for $\omega_{1d}\oplus\omega_{2\delta}\wedge dt$:

$$(3.18)\quad \omega_{1d}\oplus\omega_{2\delta}\wedge dt\;=\;\oplus_k\Big(\frac{1}{\sqrt{\lambda_k^{p-1}}}h_{1k}d_N^{p-1}\tau_{3k}\oplus(-1)^p h_{2k}\tau_{3k}\wedge dt\Big).$$

We will call p-form of type III any p-form $\omega\in L^2_p((c,+\infty)\times N)$ such that

$$\omega=\frac{1}{\sqrt{\lambda}}h_1(t)d_N^{p-1}\tau_3\oplus_M(-1)^p h_2(t)\tau_3\wedge dt,$$

where τ_3 is a normalized coclosed $(p-1)$-eigenform of Δ_N^{p-1}, corresponding to the eigenvalue λ. A direct computation shows that

$$(3.19)\quad \Delta^p_{M3}\left(\frac{1}{\sqrt{\lambda}}h_1(t)d_N^{p-1}\tau_3\oplus_M(-1)^p h_2(t)\tau_3\wedge dt\right)=$$

$$=\left(\Delta_{1\lambda}h_1+\frac{1}{f(t)}\frac{1}{g(t)}\frac{\partial g}{\partial t}\sqrt{\lambda}h_2\right)\left(\frac{1}{\sqrt{\lambda}}d_N^{p-1}\tau_3\right)$$

$$\oplus\left(\Delta_{2\lambda}h_2+\frac{1}{g^2(t)}\frac{\partial g}{\partial t}\sqrt{\lambda}h_1\right)\left((-1)^p\tau_3\wedge dt\right);$$

moreover, if $\omega=\frac{1}{\sqrt{\lambda}}h_1(t)d_N^{p-1}\tau_3\oplus_M(-1)^p h_2(t)\tau_3\wedge dt$, then

$$\|\omega\|^2_{L^2_p((c,+\infty)\times N)}=\int_c^{+\infty}g(s)^{\frac{n-2p-1}{2}}f(s)^{\frac{1}{2}}h_1(s)^2\,ds$$

$$+\int_c^{+\infty}g(s)^{\frac{n+1-2p}{2}}f(s)^{-\frac{1}{2}}h_2(s)^2\,ds.$$

Hence, introducing the transformation

$$(3.20)\qquad \begin{aligned} w_1(t)=&\ \ g^{\frac{n-2p-1}{4}}(t)f^{\frac{1}{4}}(t)h_1(t)\\ w_2(t)=&\ \ g^{\frac{n-2p+1}{4}}(t)f^{-\frac{1}{4}}(t)h_2(t),\end{aligned}$$

we find that $(\Delta^p_{M3})^F$ is unitarily equivalent to the direct sum, over $k \in \mathbb{N}$, of the Friedrichs extensions $(D_{3\lambda_k^{p-1}})^F$ of the operators

$$D_{3\lambda_k^{p-1}} : C_c^\infty(c,+\infty) \oplus C_c^\infty(c,+\infty) \longrightarrow L^2(c,+\infty) \oplus L^2(c,+\infty)$$

$$D_{3\lambda_k^{p-1}}(w_1 \oplus w_2) = \left(D_{1\lambda_k^{p-1}} w_1 + g(t)^{-\frac{3}{2}} f(t)^{-\frac{1}{2}} \frac{\partial g}{\partial t} \sqrt{\lambda_k^{p-1}} w_2 \right) \oplus \oplus \left(D_{2\lambda_k^{p-1}} w_2 + g(t)^{-\frac{3}{2}} f(t)^{-\frac{1}{2}} \frac{\partial g}{\partial t} \sqrt{\lambda_k^{p-1}} w_1 \right). \tag{3.21}$$

Hence the study of the spectrum of Δ^p_M can be reduced to the analysis of the spectral properties of the selfadjoint operators $(D_{1\lambda_k^p})^F$, $(D_{2\lambda_k^{p-1}})^F$ and $(D_{3\lambda_k^{p-1}})^F$. Since the Hodge $*$ operator maps p-forms of type I isometrically onto $(n-p)$-forms of type II, it suffices to consider the cases $i = 1$ and $i = 3$.

We have that, for $i = 1, 3$

$$\sigma_{\text{ess}}((\Delta^p_{Mi})^F) \supset \bigcup_k \sigma_{\text{ess}}((D_{i\lambda_k})^F).$$

However, since the direct sums in (3.9) and (3.18) have an infinite number of summands, we cannot conclude that

$$\sigma_{\text{ess}}((\Delta^p_{Mi})^F) = \bigcup_k \sigma_{\text{ess}}((D_{i\lambda_k})^F)$$

without additional information on the isolated eigenvalues of $(D_{i\lambda_k})^F$.

4. The case $a = -1$

For $a = -1$, the operators $D_{1\lambda_k^p}$, $D_{2\lambda_k^{p-1}}$, $D_{3\lambda_k^{p-1}}$ are given by

$$D_{1\lambda_k^p} w = -\frac{\partial^2 w}{\partial t^2} + \left[\left(\frac{n-2p-1}{2} \right)^2 b^2 + \lambda_k^p e^{2bt} \right] w,$$

$$D_{2\lambda_k^{p-1}} w = -\frac{\partial^2 w}{\partial t^2} + \left[\left(\frac{n-2p+1}{2} \right)^2 b^2 + \lambda_k^{p-1} e^{2bt} \right] w,$$

$$D_{3\lambda_k^{p-1}}(w_1 \oplus w_2) =$$

$$= -\frac{\partial^2 w_1}{\partial t^2} + \left[\left(\frac{n-2p-1}{2} \right)^2 b^2 + \lambda_k^{p-1} e^{2bt} \right] w_1 - 2be^{bt} \sqrt{\lambda_k^{p-1}} w_2 \oplus$$

$$\oplus - \frac{\partial^2 w_2}{\partial t^2} + \left[\left(\frac{n-2p+1}{2} \right)^2 b^2 + \lambda_k^{p-1} e^{2bt} \right] w_2 - 2be^{bt} \sqrt{\lambda_k^{p-1}} w_1.$$

The behaviour of the potential part depends strongly on the sign of b, hence we will investigate separately the case $b < 0$, $b = 0$, $b > 0$.

4.1. **The case $b < 0$.** For $a = -1$, $b < 0$, the situation is similar to the asymptotically hyperbolic case treated in [1]. The volume of M is infinite, and the sectional curvatures $K_M(\frac{\partial}{\partial t}, \frac{\partial}{\partial \theta_i})$, $K_M(\frac{\partial}{\partial \theta_i}, \frac{\partial}{\partial \theta_j})$ tend to $-b^2$ as $t \to +\infty$.

First of all, we will compute the essential spectrum of $(\Delta^p_{M1})^F$. To this purpose, let us observe that for every $k \in \mathbb{N}$, $\mathcal{D}((D_{1\lambda^p_k})^F) \subset W^{1,2}_0(c, +\infty)$. Indeed, if $\{w_n\}$ is a Cauchy sequence in $C^\infty_c(c, +\infty)$ with respect to the graph norm, that is, if for every $\epsilon > 0$ there exists $\bar{n}$ such that for every $n, m > \bar{n}$

$$\|w_n - w_m\|_{L^2(c,+\infty)} + \|D_{1\lambda^p_k}(w_n - w_m)\|_{L^2(c,+\infty)} < \epsilon,$$

then for every $n, m > \bar{n}$

$$\int_c^{+\infty} \left(1 + \left(\frac{n-2p-1}{2}\right)^2 b^2 + \lambda^p_k e^{2bs}\right) (w_n - w_m)^2 \, ds +$$
$$+ \int_c^{+\infty} \left(\frac{\partial}{\partial s}(w_n - w_m)\right)^2 ds < \epsilon.$$

Hence, $\{w_n\}$ is a Cauchy sequence in $W^{1,2}(c, +\infty)$. As a consequence, if $w \in \mathcal{D}((D_{1\lambda^p_k})^F)$, $w \in W^{1,2}_0(c, +\infty)$.

We are now in position to prove our first result.

Lemma 4.1. *Let $a = -1$, $b < 0$. For $0 \le p \le n-1$, for every $k \in \mathbb{N}$,*

$$\sigma_{\mathrm{ess}}((D_{1\lambda^p_k})^F) = \left[\left(\frac{n-2p-1}{2}\right)^2 b^2, +\infty\right).$$

Proof. Let us consider the Friedrichs extension $(D_{10})^F$ of the operator

$$D_{10} : C^\infty_c(c, +\infty) \longrightarrow L^2(c, +\infty)$$
$$D_{10} w = -\frac{\partial^2 w}{\partial t^2} + \left(\frac{n-2p-1}{2}\right)^2 b^2 w.$$

It is not difficult to check that, for every $k \in \mathbb{N}$, $(D_{1\lambda^p_k})^F - (D_{10})^F$ is a relatively compact perturbation of $(D_{10})^F$.

Indeed,

$$\mathcal{D}((D_{10})^F) \subseteq \mathcal{D}((D_{1\lambda^p_k})^F - (D_{10})^F) = \mathcal{D}((D_{10})^F),$$

since $X_{D_{1\lambda^p_k}} = X_{D_{10}}$ and $\mathcal{D}(D^*_{1\lambda^p_k}) = \mathcal{D}(D^*_{10})$.

Moreover, if $\{w_n\} \subset \mathcal{D}((D_{01})^F)$ satisfies the condition

$$\|w_n\|_{L^2(c,+\infty)} + \|(D_{01})^F w_n\|_{L^2(c,+\infty)} \leq C,$$

$\{w_n\}$ is bounded in $W^{1,2}(c,+\infty)$; hence it is bounded in $L^\infty(c,+\infty)$ and in $W^{1,2}(K)$ for every compact subset $K \subset (c,+\infty)$. Thus, since for $b<0$ $e^{2bt} \in L^2(c,+\infty) \cap L^\infty(c,+\infty)$, for every $\tilde{c} > c$ and for every $n,m \in \mathbb{N}$

$$\int_c^{+\infty} (\lambda_k^p)^2 e^{4bs}(w_n - w_m)^2\, ds \leq$$

$$\leq C_1 \int_c^{\tilde{c}} (w_n - w_m)^2\, ds + C_2 \int_{\tilde{c}}^{+\infty} (\lambda_k^p)^2 e^{4bs}\, ds,$$

with C_1, C_2 not depending on $\tilde{c}$. Let us consider a sequence $\{c_h\} \subset (c,+\infty)$ such that $c_h \to +\infty$ as $h \to +\infty$ and for every $h \in \mathbb{N}$

$$\int_{c_h}^{+\infty} (\lambda_k^p)^2 e^{4bs}\, ds < \frac{1}{h}.$$

For $h=1$, thanks to the Rellich-Kondrachov theorem, there exists a subsequence $\{w_{n(1)}\} \subseteq \{w_n\}$ which converges in $L^2(c,\tilde{c}_1)$. Hence, for every $\eta > 0$ there exists $\bar{n}(1)$ such that for every $n,m > \bar{n}(1)$

$$\int_c^{+\infty} (\lambda_k^p)^2 e^{4bs}(w_{n(1)} - w_{m(1)})^2\, ds < \eta + 1.$$

Analogously, for $h=2$ there exists a subsequence $\{w_{n(2)}\} \subseteq \{w_{n(1)}\}$ such that for every $\eta > 0$ there exists $\bar{n}(2)$ for which

$$n,m > \bar{n}(2) \Longrightarrow \int_c^{+\infty} (\lambda_k^p)^2 e^{4bs}(w_{n(2)} - w_{m(2)})^2\, ds < \eta + \frac{1}{2}.$$

Iterating this argument, for every $h \in \mathbb{N}$ we find a subsequence $\{w_{n(h)}\} \subseteq \{w_{n(h-1)}\}$ such that for every $\eta > 0$ there exists $\bar{n}(h)$ for which

$$n,m > \bar{n}(h) \Longrightarrow \int_c^{\infty} (\lambda_k^p)^2 e^{4bs}(w_{n(h)} - w_{m(h)})^2\, ds < \eta + \frac{1}{h}.$$

Through a Cantor diagonal process, we can then find a subsequence $\{w_{n_l}\} \subset \{w_n\}$, such that for every $\eta > 0$ and every $h \in \mathbb{N}$, if $l,m > \bar{l} = \bar{n}(h)$

$$\|((D_{1\lambda_k^p})^F - (D_{10})^F)(w_{n_l} - w_{n_m})\|^2_{L^2(c,+\infty)} =$$

$$= \int_c^{+\infty} (\lambda_k^p)^2 e^{4bs}(w_{n_l} - w_{n_m})\, ds < \eta + \frac{1}{h}.$$

Hence, $(D_{1\lambda_k^p})^F - (D_{10})^F$ is a relatively compact perturbation of $(D_{10})^F$.

Since $\sigma_{\mathrm{ess}}(D_{10}^F) = [(\frac{n-2p-1}{2})^2 b^2, +\infty)$, the conclusion follows. $\square$

As a consequence, $\left[\left(\frac{n-2p-1}{2}\right)^2 b^2, +\infty\right) \subseteq \sigma_{\mathrm{ess}}((\Delta^p_{M1})^F)$. On the other hand,

Lemma 4.2. *Let $a = -1$, $b < 0$. Then, if $\mu < \left(\frac{n-2p-1}{2}\right)^2 b^2$, $\mu \notin \sigma_{\mathrm{ess}}((\Delta^p_{M1})^F)$.*

Proof. If $\mu \in \sigma_{\mathrm{ess}}((\Delta^p_{M1})^F)$, there exists a Weyl sequence $\{\omega_k\} \subset \mathcal{D}((\Delta^p_{M1})^F)$ for μ. $\{\omega_k\}$ has no convergent subsequence and we have

$$\langle \omega_k, \omega_k \rangle_{L^2_p((c,+\infty)\times N)} \leq C,$$

$$\lim_{k\to+\infty} ((\Delta^p_{M1})^F \omega_k - \mu\omega_k) = 0.$$

Moreover, we can suppose that

$$\omega_k = h_{\lambda^p_k} \tau_{\lambda^p_k},$$

where, for every $k \in \mathbb{N}$, $\tau_{\lambda^p_k}$ is a normalized coclosed p-eigenform of Δ^p_N corresponding to the eigenvalue λ^p_k and $\lambda^p_k \to +\infty$ as $k \to +\infty$.

Hence, there exists a bounded sequence $\{w_k\}$ in $L^2(c, +\infty)$ such that, for every $k \in \mathbb{N}$, $w_k \in \mathcal{D}((D_{1\lambda^p_k})^F)$, and such that

$$\lim_{k\to+\infty} \|(D_{1\lambda^p_k})^F w_k - \mu w_k\|_{L^2(c,+\infty)} = 0,$$

from which we cannot extract any convergent subsequence.

Then

$$\langle (D_{1\lambda^p_k})^F w_k - \mu w_k, w_k \rangle_{L^2(c,+\infty)} \longrightarrow 0$$

as $k \to +\infty$ and since for every $k \in \mathbb{N}$

$$\mathcal{D}((D_{1\lambda^p_k})^F) \subset W^{1,2}_0(c, +\infty),$$

then

$$\int_c^{+\infty} \left(\frac{\partial w_k}{\partial s}\right)^2 ds + \int_c^{+\infty} \lambda^p_k e^{2bs} w_k^2 \, ds +$$
$$+ \int_c^{+\infty} \left(\left(\frac{n-2p-1}{2}\right)^2 b^2 - \mu\right) w_k^2 \, ds \longrightarrow 0$$

as $k \to +\infty$. Thus

$$\int_c^{+\infty} w_k^2 \, ds \longrightarrow 0$$

as $k \to +\infty$, and we get a contradiction. Hence, if $\mu < \left(\frac{n-2p-1}{2}\right)^2 b^2$, μ cannot belong to the essential spectrum of $(\Delta^p_{M1})^F$. $\square$

As a consequence, we have the following

Proposition 4.3. *Let $a = -1$, $b < 0$. For $0 \leq p \leq n-1$,*

$$\sigma_{\mathrm{ess}}((\Delta^p_{M1})^F) = \left[\left(\frac{n-2p-1}{2}\right)^2 b^2, +\infty\right).$$

By duality,

Proposition 4.4. *Let $a = -1$, $b < 0$. For $1 \leq p \leq n$,*

$$\sigma_{\mathrm{ess}}((\Delta^p_{M2})^F) = \left[\left(\frac{n-2p+1}{2}\right)^2 b^2, +\infty\right).$$

We still have to investigate the spectrum of $(\Delta^p_{M3})^F$. First of all, we compute the essential spectrum of $(D_{3\lambda_k^{p-1}})^F$ for every $k \in \mathbb{N}$.

Lemma 4.5. *Let $a = -1$, $b < 0$. For $1 \leq p \leq n-1$ and for every $k \in \mathbb{N}$,*

$$\sigma_{\mathrm{ess}}((D_{3\lambda_k^{p-1}})^F) = \tag{4.1}$$
$$= \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right).$$

Proof. Let us consider the Friedrichs extension $(D_{30})^F$ of the operator

$$D_{30} : C^\infty_c(c, +\infty) \oplus C^\infty_c(c, +\infty) \longrightarrow L^2(c, +\infty) \oplus L^2(c, +\infty)$$
$$D_{30}(w_1 \oplus w_2) = -\frac{\partial^2 w_1}{\partial t^2} + \left(\frac{n-2p-1}{2}\right)^2 b^2 w_1 \oplus$$
$$\oplus -\frac{\partial^2 w_2}{\partial t^2} + \left(\frac{n-2p+1}{2}\right)^2 b^2 w_2.$$

Since the essential spectrum of $(D_{30})^F$ is equal to

$$\left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right),$$

it suffices to show that $(D_{3\lambda_k^{p-1}})^F - (D_{30})^F$ is a relatively compact perturbation of $(D_{30})^F$.

Now,

$$\mathcal{D}((D_{30})^F) \subseteq \mathcal{D}((D_{3\lambda_k^{p-1}})^F - (D_{30})^F) = \mathcal{D}((D_{30})^F),$$

since $X_{D_{30}} = X_{D_{3\lambda_k^{p-1}}}$ and $\mathcal{D}(D^*_{30}) = \mathcal{D}(D^*_{3\lambda_k^{p-1}})$.

Moreover, if the sequence $\{w_{1n} \oplus w_{2n}\}$ is such that, for every $n \in \mathbb{N}$,

$$\|(w_{1n} \oplus w_{2n})\|_{L^2(c,+\infty)\oplus L^2(c,+\infty)} + \|D^F_{30}(w_{1n} \oplus w_{2n})\|_{L^2(c,+\infty)\oplus L^2(c,+\infty)} \leq C,$$

then $\{w_{1n}\}$ and $\{w_{2n}\}$ are bounded in $W^{1,2}(c,+\infty)$; hence, they are bounded also in $L^\infty(c,+\infty)$ and in $W^{1,2}(K)$ for any compact subset $K\subset(c,+\infty)$.

Since $b<0$, $e^{2bt}\in L^2(c,+\infty)\cap L^\infty(c,+\infty)$ and $e^{bt}\in L^2(c,+\infty)\cap L^\infty(c,+\infty)$; thus, arguing as in the proof of Lemma 4.1, we can construct a subsequence $\{w_{1n_h}\oplus w_{2n_h}\}$ such that for every $\delta>0$ there exists $\bar h$ such that whenever $h,l>\bar h$

$$\|((D_{3\lambda_k^{p-1}})^F-(D_{30})^F)((w_{1n_h}-w_{1n_l})\oplus(w_{2n_h}-w_{2n_l}))\|_{L^2(c,+\infty)\oplus L^2(c,+\infty)}=$$

$$=\|\lambda_k^{p-1}e^{2bt}(w_{1n_h}-w_{1n_l})-2b\sqrt{\lambda_k^{p-1}}e^{bt}(w_{2n_h}-w_{2n_l})\|_{L^2(c,+\infty)}+$$

$$+\|\lambda_k^{p-1}e^{2bt}(w_{2n_h}-w_{2n_l})-2b\sqrt{\lambda_k^{p-1}}e^{bt}(w_{1n_h}-w_{1n_l})\|_{L^2(c,+\infty)}<\delta.$$

Hence, $(D_{3\lambda_k^{p-1}})^F-(D_{30})^F$ is a relatively compact perturbation of $(D_{30})^F$, and $\sigma_{\rm ess}((D_{3\lambda_k^{p-1}})^F)=\sigma_{\rm ess}((D_{30})^F)$. □

On the other hand,

Lemma 4.6. *Let $a=-1$, $b<0$. If*

$$0<\mu<\min\left\{\left(\frac{n-2p-1}{2}\right)^2b^2,\left(\frac{n-2p+1}{2}\right)^2b^2\right\},$$

then $\mu\notin\sigma_{\rm ess}((\Delta^p_{M3})^F)$.

Proof. Let

$$0<\mu<\min\left\{\left(\frac{n-2p-1}{2}\right)^2b^2,\left(\frac{n-2p+1}{2}\right)^2b^2\right\};$$

then $\mu\in\sigma_{\rm ess}((\Delta^p_{M3})^F)$ if and only if there exist a sequence $\{\mu_k\}$ of eigenvalues of $(\Delta^p_{M3})^F$ and a corresponding sequence of normalized, mutually orthogonal, eigenforms $\{\Phi_k\}$ of $(\Delta^p_{M3})^F$ such that for every $k\in\mathbb{N}$

$$(\Delta^p_{M3})^F\Phi_k-\mu_k\Phi_k=0$$

and

$$\mu_k\longrightarrow\mu\qquad\text{as }k\to+\infty.$$

In view of the weak Kodaira decomposition, replacing $\{\Phi_k\}$ by a subsequence (again denoted by the same symbol for shortness), we can suppose that either $\delta^p_M\Phi_k=0$ for every $k\in\mathbb{N}$, or $d^p_M\Phi_k=0$ for every $k\in\mathbb{N}$.

a) In the first case, since $\mu\neq0$, for every $k\in\mathbb{N}$ $d^p_M\Phi_k\neq0$; we have $\|d^p_M\Phi_k\|_{L^2_{p+1}((c,+\infty)\times N)}<C$, and

$$\Delta^{p+1}_M d^p_M\Phi_k-\mu\, d^p_M\Phi_k\longrightarrow0$$

as $k \to +\infty$ because

$$\|\Delta_M^{p+1} d_M^p \Phi_k - \mu\, d_M^p \Phi_k\|_{L^2_{p+1}((c,+\infty)\times N)} \leq$$

$$\|\Delta_M^{p+1} d_M^p \Phi_k - \mu_k\, d_M^p \Phi_k\|_{L^2_{p+1}((c,+\infty)\times N)} + |\mu_k - \mu| \|d_M^p \Phi_k\|_{L^2_{p+1}((c,+\infty)\times N)},$$

where

$$\Delta_M^{p+1} d_M^p \Phi_k - \mu_k\, d_M^p \Phi_k = 0$$

for every $k \in \mathbb{N}$, and $\mu_k \to \mu$ as $k \to +\infty$. Moreover, $\{d_M^p \Phi_k\}$ has no convergent subsequences, since for $i \neq j$ $\langle d_M^p \Phi_i, d_M^p \Phi_j \rangle_{L^2_{p+1}((c,+\infty)\times N)} = 0$ and $\|d_M^p \Phi_k\|^2 = \mu_k > \mu - \epsilon$ for k big enough.

Now, let $\psi \in C_c^\infty(M)$ be such that

$$0 \leq \psi(x) \leq 1 \quad \text{for every } x \in M,$$

$$\psi(x) = 1 \quad \text{for } x \in M \setminus ((c, +\infty) \times N)$$

and

$$\psi(x) = 0 \quad \text{for } x \in (2c, +\infty) \times N.$$

If we set, for every $k \in \mathbb{N}$,

$$\tilde{\omega}_k := (1 - \psi)(d_M^p \Phi_{2k+1} - d_M^p \Phi_{2k}),$$

arguing as in the proof of Satz 3.1 in [5] we find that $\{\tilde{\omega}_k\}$ is a Weyl sequence for μ for the operator $(\Delta_M^{p+1})^F$. Since an explicit computation shows that, for every $k \in \mathbb{N}$, $d_M^p \Phi_k$ is a $(p+1)$-form of type II and $(1-\psi) d_M^p \Phi_k \in \mathcal{D}((\Delta_{M2}^{p+1})^F)$, we can argue that μ lies in the essential spectrum of $(\Delta_{M2}^{p+1})^F$. But this is not possible because by assumption

$$\mu < \left(\frac{n - 2p - 1}{2}\right)^2 b^2 = \left(\frac{n - 2(p+1) + 1}{2}\right)^2 b^2.$$

b) Consider now the second case. Since $\mu > 0$, for every $k \in \mathbb{N}$ $\delta_M^p \Phi_k \neq 0$; following the same argument as in part a), if we set for every $k \in \mathbb{N}$

$$\tilde{\omega}_k := (1 - \psi)(\delta_M^p \Phi_{2k+1} - \delta_M^p \Phi_{2k}),$$

where $\psi(x) \in C_c^\infty(M)$ is chosen as in part a), we find that $\{\tilde{\omega}_k\}$ is a Weyl sequence for μ for $(\Delta_M^{p-1})^F$. Since an explicit computation shows that, for every $k \in \mathbb{N}$, $\delta_M^p \Phi_k$ is a $(p-1)$-form of type I, we conclude that μ lies in the essential spectrum of $(\Delta_{M1}^{p-1})^F$. But this is not possible since by assumption

$$\mu < \left(\frac{n - 2p + 1}{2}\right)^2 b^2 = \left(\frac{n - 2(p-1) - 1}{2}\right)^2 b^2.$$

$\square$

In the proof of Lemma 4.6, it is essential that $\{\mu_k\}$ is a sequence of strictly positive real numbers. Hence, by this technique we cannot determine whether 0 is an isolated eigenvalue of infinite multiplicity of $(\Delta^p_{M3})^F$ or not.

As a consequence, all we can state about the essential spectrum of $(\Delta^p_{M3})^F$ is:

Proposition 4.7. *Let $a=-1$, $b<0$. Then, for $1\le p\le n-1$,*

$$\sigma_{\mathrm{ess}}((\Delta^p_{M3})^F)\setminus\{0\} = \tag{4.2}$$
$$= \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right).$$

Combining the results of Proposition 4.3, Proposition 4.4 and Proposition 4.7, finally we can state the following

Theorem 4.8. *Let $a=-1$, $b<0$, $0\le p\le n$. Then, if $p\neq\frac{n\pm1}{2}$*

$$\sigma_{\mathrm{ess}}(\Delta^p_M)\setminus\{0\} = \tag{4.3}$$
$$= \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right),$$

whilst if $p=\frac{n+1}{2}$ or $p=\frac{n-1}{2}$

$$\sigma_{\mathrm{ess}}(\Delta_M) = [0,+\infty).$$

4.2. **The case $b=0$.** For $a=-1$, $b=0$, the operators $D_{1\lambda^p_k}$, $D_{2\lambda^{p-1}_k}$, $D_{3\lambda^{p-1}_k}$ are simply

$$D_{1\lambda^p_k} w = -\frac{\partial^2 w}{\partial t^2} + \lambda^p_k w,$$
$$D_{2\lambda^{p-1}_k} w = -\frac{\partial^2 w}{\partial t^2} + \lambda^{p-1}_k w,$$
$$D_{3\lambda^{p-1}_k}(w_1\oplus w_2) = -\frac{\partial^2 w_1}{\partial t^2} + \lambda^{p-1}_k w_1 \oplus -\frac{\partial^2 w_2}{\partial t^2} + \lambda^{p-1}_k w_2;$$

hence the essential spectrum of $D^F_{1\lambda^p_k}$ (resp. $D^F_{3\lambda^{p-1}_k}$) is equal to $[\lambda^p_k,+\infty)$ (resp. $[\lambda^{p-1}_k,+\infty)$). Moreover, an explicit computation shows that, for every $k\in\mathbb{N}$, $(D_{1\lambda^p_k})^F$ (resp. $(D_{3\lambda^{p-1}_k})^F$) has no eigenvalues. As a consequence, we recover the well-known result for cylindrical ends (see e.g. [8]):

Theorem 4.9. *Let $a=-1$, $b=0$. Then, for $0\le p\le n$, the essential spectrum of Δ_M^p is given by*

$$\sigma_{\mathrm{ess}}(\Delta_M^p)=\bigcup_k([\lambda_k^p,+\infty)\cup[\lambda_k^{p-1},+\infty))=[\overline{\lambda},+\infty),$$

where $\overline{\lambda}=\min_k\left\{\lambda_k^p,\lambda_k^{p-1}\right\}$.

In particular, if the p-th or the $(p-1)$-th Betti number of N does not vanish, $\overline{\lambda}=0$, otherwise $\overline{\lambda}>0$.

4.3. **The case $b>0$.** As in the previous cases, we begin with the spectral analysis of $D^F_{1\lambda_k^p}$ for every $k\in\mathbb{N}$:

Lemma 4.10. *Let $a=-1$, $b>0$. Then, for every $k\in\mathbb{N}$, if $\lambda_k^p=0$, $\sigma_{\mathrm{ess}}((D_{1\lambda_k^p})^F)=\left[\left(\frac{n-2p-1}{2}\right)^2b^2,+\infty\right)$; if, on the contrary, $\lambda_k^p>0$, $\sigma_{\mathrm{ess}}((D_{1\lambda_k^p})^F)=\emptyset$.*

Proof. The first assertion is obvious; as for the other, it is well-known (see e.g. [9], Thm. 3.13) that the spectrum of any selfadjoint extension of an operator of type $-\frac{\partial^2}{\partial t^2}+V(t)$ acting on $C_c^\infty(c,+\infty)$ is purely discrete if, and only if, for every $h\in(0,1)$

$$\lim_{t\to+\infty}\int_t^{t+h}V(s)\,ds=+\infty.$$

Since for $\lambda_k^p>0$

$$\int_t^{t+h}\left[\left(\frac{n-2p-1}{2}\right)^2b^2+\lambda_k^pe^{2bs}\right]ds=$$

$$=\left(\frac{n-2p-1}{2}\right)^2b^2h+\frac{\lambda_k^p}{2b}e^{2bt}(e^{2bh}-1)\longrightarrow+\infty$$

as $t\to+\infty$, the conclusion follows. $\square$

Now, an easy computation shows that for every $k\in\mathbb{N}$
(4.4)

$$\langle(D_{1\lambda_k^p})^Fw,w\rangle_{L^2(c,+\infty)}\ge\left(\left(\frac{n-2p-1}{2}\right)^2b^2+\lambda_k^p\right)\langle w,w\rangle_{L^2(c,+\infty)}.$$

As a consequence, $\sigma_p((\Delta_{M1}^p)^F)$ have no cluster points and every eigenvalue of $(\Delta_{M1}^p)^F$ has finite multiplicity. Hence:

Proposition 4.11. *Let $a=-1$, $b>0$. For $0\leq p\leq n-1$, if the p-th Betti number of N vanishes, $\sigma_{\mathrm{ess}}((\Delta^p_{M1})^F)=\emptyset$; if, on the contrary, the p-th Betti number of N is different from zero, $\sigma_{\mathrm{ess}}((\Delta^p_{M1})^F)=\left[\left(\frac{n-2p-1}{2}\right)^2 b^2,+\infty\right)$.*

By duality,

Proposition 4.12. *Let $a=-1$, $b>0$. For $1\leq p\leq n$, if the $(p-1)$-th Betti number of N vanishes, $\sigma_{\mathrm{ess}}((\Delta^p_{M2})^F)=\emptyset$; if, on the contrary, the $(p-1)$-th Betti number of N is different from zero, $\sigma_{\mathrm{ess}}((\Delta^p_{M2})^F)=\left[\left(\frac{n-2p+1}{2}\right)^2 b^2,+\infty\right)$.*

We shall now investigate the spectrum of $(\Delta^p_{M3})^F$. As a first step, we will compute the essential spectrum of $(D_{3\lambda^{p-1}_k})^F$ for every $k\in\mathbb{N}$. To this purpose, we need a preliminary Lemma:

Lemma 4.13. *For every $K\in\mathbb{R}$ and for every $\lambda\geq 0$, the essential spectrum of the Friedrichs extension D^F of the operator*

$$D: C^\infty_c(c,+\infty)\oplus C^\infty_c(c,+\infty)\longrightarrow L^2(c,+\infty)\oplus L^2(c,+\infty)$$

$$\mathcal{D}(w_1\oplus w_2)=-\frac{\partial^2 w_1}{\partial t^2}+\left[K+\lambda e^{2bt}\right]w_1-2b\sqrt{\lambda}e^{bt}w_2\oplus$$

$$\oplus-\frac{\partial^2 w_2}{\partial t^2}+\left[K+\lambda e^{2bt}\right]w_2-2b\sqrt{\lambda}e^{bt}w_1$$

is empty.

Proof. First of all, through an argument similar to that of Satz 3.1 in [5], it can be shown that the essential spectrum of D^F does not depend on the choice of the first endpoint c of $(c,+\infty)$. Hence, given K and $\mu\geq 0$ we can suppose that for every $t>c$

$$K-\mu+\lambda e^{2bt}\pm 2be^{bt}\sqrt{\lambda}>C>0. \tag{4.5}$$

Consider now the closed subspaces $\mathcal{V}_1$, $\mathcal{V}_2$ of $L^2(c,+\infty)\oplus L^2(c,+\infty)$ defined as

$$\mathcal{V}_1:=\{w_1\oplus w_2\,|\,w_1=w_2\},$$
$$\mathcal{V}_2:=\{w_1\oplus w_2\,|\,w_1=-w_2\}.$$

They are orthogonal in $L^2(c,+\infty)\oplus L^2(c,+\infty)$ and any $w_1\oplus w_2$ can be written as

$$w_1\oplus w_2=\left(\frac{w_1+w_2}{2}\oplus\frac{w_1+w_2}{2}\right)+\left(\frac{w_1-w_2}{2}\oplus\frac{w_2-w_1}{2}\right).$$

Hence,

$$L^2(c,+\infty)\oplus L^2(c,+\infty)=\mathcal{V}_1\oplus\mathcal{V}_2. \tag{4.6}$$

Moreover, an explicit computation shows that the operator D is invariant under decomposition (4.6). As a consequence, D^F splits as

$$D^F = (D_{|\mathcal{V}_1})^F \oplus (D_{|\mathcal{V}_2})^F,$$

where, for $i = 1, 2$, $D_{|\mathcal{V}_i}$ is the restriction of D to

$$(C_c^\infty(c, +\infty) \oplus C_c^\infty(c, +\infty)) \cap \mathcal{V}_i;$$

thus,

$$\sigma_{\text{ess}}(D^F) = \sigma_{\text{ess}}(D^F_{|\mathcal{V}_1}) \cup \sigma_{\text{ess}}(D^F_{|\mathcal{V}_2}).$$

Let us begin with $D^F_{|\mathcal{V}_1}$. If $\mu \in \sigma_{\text{ess}}(D^F_{|\mathcal{V}_1})$, there exists a Weyl sequence for μ, that is a sequence $\{w_n\} \subset \mathcal{D}(D^F_{|\mathcal{V}_1})$ such that $\|w_n\|_{L^2(c,+\infty)} \leq C$ and $(D - \mu)(w_n \oplus w_n) \longrightarrow 0$ as $n \to +\infty$, with no convergent subsequences. Then

$$\langle (D - \mu)(w_n \oplus w_n), (w_n \oplus w_n)\rangle_{L^2(c,+\infty)\oplus L^2(c,+\infty)} \longrightarrow 0$$

as $n \to +\infty$. Thus

$$\int_c^{+\infty} \left(\frac{\partial w_n}{\partial s}\right)^2 ds + \int_c^{+\infty} \left[K - \mu + \lambda e^{2bs} - 2b\sqrt{\lambda} e^{bs}\right] w_n^2 \, ds \longrightarrow 0$$

as $n \to +\infty$. Since (4.5) implies that

$$\int_c^{+\infty} w_n^2 \, ds < C^{-1} \int_c^{+\infty} \left[K - \mu + \lambda e^{2bs} - 2b\sqrt{\lambda} e^{bs}\right] w_n^2 \, ds,$$

we find that $\|w_n\|_{L^2(c,+\infty)} \to 0$ as $n \to +\infty$. Hence, $\sigma_{\text{ess}}(D^F_{|\mathcal{V}_1}) = \emptyset$.

Suppose now that $\mu \in \sigma_{\text{ess}}(D^F_{|\mathcal{V}_2})$. If $\{w_n\}$ is a Weyl sequence for μ for $D^F_{|\mathcal{V}_2}$, we have

$$\langle (D - \mu)(w_n \oplus -w_n), (w_n \oplus -w_n)\rangle_{L^2(c,+\infty)\oplus L^2(c,+\infty)} \longrightarrow 0$$

as $n \to +\infty$. Hence

$$\int_c^{+\infty} \left(\frac{\partial w_n}{\partial s}\right)^2 ds + \int_c^{+\infty} \left[K - \mu + \lambda e^{2bs} + 2b\sqrt{\lambda} e^{bs}\right] w_n^2 \, ds \longrightarrow 0$$

as $n \to +\infty$. In view of (4.5), $\|w_n\|_{L^2(c,+\infty)} \to 0$ as $n \to +\infty$. But a Weyl sequence cannot converge. Hence, $\sigma_{\text{ess}}(D^F_{|\mathcal{V}_2}) = \emptyset$. □

We can now compute the essential spectrum of $(D_{3\lambda_k^{p-1}})^F$:

Lemma 4.14. *Let $a = -1$, $b > 0$. Then, for $1 \leq p \leq n-1$ and for any $k \in \mathbb{N}$,*

$$\sigma_{\text{ess}}((D_{3\lambda_k^{p-1}})^F) = \emptyset.$$

Proof. Let us consider the Friedrichs extension $(D_{30})^F$ of the operator

$$D_{30} : C_c^\infty(c,+\infty) \oplus C_c^\infty(c,+\infty) \longrightarrow L^2(c,+\infty) \oplus L^2(c,+\infty)$$

$$\begin{aligned} D_{30}(w_1 \oplus w_2) = &-\frac{\partial^2 w_1}{\partial t^2} + \tilde{K} w_1 + \lambda_k^{p-1} e^{2bt} w_1 - 2b\sqrt{\lambda_k^{p-1}} e^{bt} w_2 \oplus \\ &\oplus -\frac{\partial^2 w_2}{\partial t^2} + \tilde{K} w_2 + \lambda_k^{p-1} e^{2bt} w_2 - 2b\sqrt{\lambda_k^{p-1}} e^{bt} w_1, \end{aligned} \tag{4.7}$$

where

$$\tilde{K} = \max\left\{ \left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2 \right\}.$$

From the previous Lemma, we know that $\sigma_{\text{ess}}((D_{30})^F) = \emptyset$. We will show that $((D_{3\lambda_k^{p-1}})^F - (D_{30})^F)$ is a relatively compact perturbation of $(D_{30})^F$.

First of all, since a straightforward computation shows that $X_{D_{30}} \subseteq X_{D_{3\lambda_k^{p-1}}}$ and $\mathcal{D}(D_{30}^*) \subseteq \mathcal{D}(D^*_{3\lambda_k^{p-1}})$, then

$$\mathcal{D}((D_{30})^F) \subseteq \mathcal{D}((D_{3\lambda_k^{p-1}})^F - (D_{30})^F) = \mathcal{D}((D_{3\lambda_k^{p-1}})^F).$$

We still have to show that, given a sequence $\{w_{1n} \oplus w_{2n}\}$ in $\mathcal{D}((D_{30})^F)$ such that

$$\|w_{1n} \oplus w_{2n}\|^2_{L^2(c,+\infty)} + \|((D_{30})^F)(w_{1n} \oplus w_{2n})\|^2_{L^2(c,+\infty)} \leq C$$

there exists a subsequence $\{w_{1n_k} \oplus w_{2n_k}\}$ such that

$$\left\{((D_{3\lambda_k^{p-1}})^F - (D_{30})^F)(w_{1n_k} \oplus w_{2n_k})\right\}$$

converges.

Now, the fact that $\|(D_{30})^F(w_{1n} \oplus w_{2n})\|^2_{L^2(c,+\infty)} \leq C$ is equivalent to the inequalities

$$\| -\frac{\partial^2 w_{1n}}{\partial t^2} + \left[\tilde{K} + \lambda_k^{p-1} e^{2bt}\right] w_{1n} - 2be^{bt}\sqrt{\lambda_k^{p-1}} w_{2n}\|_{L^2(c,+\infty)} \leq C, \tag{4.8}$$

$$\| -\frac{\partial^2 w_{2n}}{\partial t^2} + \left[\tilde{K} + \lambda_k^{p-1} e^{2bt}\right] w_{2n} - 2be^{bt}\sqrt{\lambda_k^{p-1}} w_{1n}\|_{L^2(c,+\infty)} \leq C, \tag{4.9}$$

which in turn imply

$$\begin{aligned} \| &-\frac{\partial^2 (w_{1n} + w_{2n})}{\partial t^2} + \left[\tilde{K} + \lambda_k^{p-1} e^{2bt}\right] (w_{1n} + w_{2n}) + \\ &- 2be^{bt}\sqrt{\lambda_k^{p-1}} (w_{2n} + w_{1n})\|_{L^2(c,+\infty)} \leq C, \end{aligned} \tag{4.10}$$

$$\text{(4.11)}\quad \| - \frac{\partial^2(w_{1n}-w_{2n})}{\partial t^2} + \left[\tilde{K} + \lambda_k^{p-1} e^{2bt}\right](w_{1n}-w_{2n}) + \\ - 2be^{bt}\sqrt{\lambda_k^{p-1}}(w_{2n}-w_{1n})\|_{L^2(c,+\infty)} \le C.$$

By taking the inner product with $(w_{1n}+w_{2n})$ and $(w_{1n}-w_{2n})$ respectively, (4.10) and (4.11) yield:

$$\text{(4.12)}\quad \int_c^{+\infty}\left(\frac{\partial(w_{1n}+w_{2n})}{\partial s}\right)^2 ds + \\ + \int_c^{+\infty}\left(\tilde{K} + \lambda_k^{p-1}e^{2bs} - 2b\sqrt{\lambda_k^{p-1}}e^{bs}\right)(w_{1n}+w_{2n})^2\, ds \le C,$$

$$\text{(4.13)}\quad \int_c^{+\infty}\left(\frac{\partial(w_{1n}-w_{2n})}{\partial s}\right)^2 ds + \\ + \int_c^{+\infty}\left(\tilde{K} + \lambda_k^{p-1}e^{2bs} + 2b\sqrt{\lambda_k^{p-1}}e^{bs}\right)(w_{1n}-w_{2n})^2\, ds \le C.$$

Now, there exists $\bar{c}$ such that for every strictly positive $\lambda_k^{p-1} \in \sigma_p(\Delta_N)$ and for every $t > \bar{c}$

$$\tilde{K} + \lambda_k^{p-1}e^{2bt} \pm 2b\sqrt{\lambda_k^{p-1}}e^{bt} > \tilde{C}e^{\frac{3}{2}bt},$$

with $\tilde{C} > 0$. Since in view of the results of [5] we can assume $c > \bar{c}$, we obtain from (4.12) and (4.13) that

$$\int_c^{+\infty} e^{\frac{3}{2}bs}(w_{1n} \pm w_{2n})^2\, ds \le C.$$

Hence, for $i = 1, 2$,

$$\int_c^{+\infty} e^{\frac{3}{2}bs}(w_{in})^2\, ds \le C.$$

Moreover, $\{w_{1n}\}$ and $\{w_{2n}\}$ are bounded in $W^{1,2}(c,+\infty)$; thus, they are bounded also in $L^\infty(c,+\infty)$ and in $W^{1,2}(K)$ for every compact subset $K \subset (c,+\infty)$. As a consequence, if we set

$$K_1(n,p) = \left(\frac{n-2p-1}{2}\right)^2 b^2,$$

$$K_2(n,p) = \left(\frac{n-2p+1}{2}\right)^2 b^2,$$

for every $n, m \in \mathbb{N}$

$$\|((D_{3\lambda_k^{p-1}})^F - (D_{30})^F)((w_{1n}-w_{1m}) \oplus (w_{2n}-w_{2m}))\|^2_{L^2(c,+\infty)} =$$

$$= \sum_{i=1,2} \int_c^{+\infty} (K_i(p,n) - \tilde{K})^2 (w_{in} - w_{im})^2 \, ds \le$$

$$\le \sum_{i=1,2} (K_i(p,n) - \tilde{K})^2 \| e^{-\frac{3}{4}bt}(w_{in} - w_{im}) \|_{L^2} \| e^{\frac{3}{4}bt}(w_{in} - w_{im}) \|_{L^2} \le$$

$$\le C \sum_{i=1,2} \| e^{-\frac{3}{4}bt}(w_{in} - w_{im}) \|_{L^2(c,+\infty)}.$$

Since

$$e^{-\frac{3}{4}bt} \in L^2(c,+\infty) \cap L^\infty(c,+\infty),$$

following the argument of Lemma 4.1, we can extract from $\{w_{1n} \oplus w_{2n}\}$ a subsequence $\{w_{1n_k} \oplus w_{2n_k}\}$ such that

$$\left\{ ((D_{3\lambda_k^{p_1}})^F - (D_{30})^F)(w_{1n_k} \oplus w_{2n_k}) \right\}$$

converges.

Hence, $\sigma_{\mathrm{ess}}(D_{3\lambda_k^{p-1}})^F) = \sigma_{\mathrm{ess}}(D_{30}^F) = \emptyset$. □

We still have to investigate the cluster points of $\sigma_p((\Delta^p_{M3})^F)$, which could be additional points in the essential spectrum of Δ^p_M. The following Lemma holds:

Lemma 4.15. *Let $a = -1$, $b > 0$. Let $0 < \mu \in \overline{\sigma_p((\Delta^p_{M3}))^F}$.*

(1) *If the p-th and the $(p-1)$-th Betti numbers of N both vanish, μ is an isolated eigenvalue of finite multiplicity;*

(2) *if the p-th Betti number of N vanishes whilst the $(p-1)$-th Betti number of N is different from zero, and if μ is not an isolated eigenvalue of finite multiplicity, then $\mu \ge \left(\frac{n-2p+1}{2}\right)^2 b^2$;*

(3) *if the p-th Betti number of N is different from zero whilst the $(p-1)$-th Betti number of N vanishes, and if μ is not an isolated eigenvalue of finite multiplicity, then $\mu \ge \left(\frac{n-2p-1}{2}\right)^2 b^2$;*

(4) *if the p-th and the $(p-1)$-th Betti numbers of N are both different from zero, and if μ is not an isolated eigenvalue of finite multiplicity, then $\mu \ge \min\left\{ \left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2 \right\}$.*

Proof. If μ is an eigenvalue of infinite multiplicity of $(\Delta^p_{M3})^F$ or is a cluster point of $\sigma_p((\Delta^p_{M3})^F)$, there exist a sequence $\{\mu_k\}$ of eigenvalues of $(\Delta^p_{M3})^F$ and a corresponding sequence of normalized, mutually orthogonal eigenforms $\{\Phi_k\}$ such that for every $k \in \mathbb{N}$

$$\Delta^p_M \Phi_k - \mu_k \Phi_k = 0$$

and

$$\mu_k \longrightarrow \mu \qquad \text{as } k \to +\infty.$$

In view of the weak Kodaira decomposition, replacing $\{\Phi_k\}$ by a subsequence (again denoted by the same symbol for shortness) we can suppose that either $\delta_M^p \Phi_k = 0$ for every $k \in \mathbb{N}$, or $d_M^p \Phi_k = 0$ for every $k \in \mathbb{N}$. Following the argument of Lemma 4.6, we see that, in the first case, μ lies in the essential spectrum of the operator $(\Delta_{M2}^{p+1})^F$, whilst, in the second case, μ lies in the essential spectrum of the operator $(\Delta_{M1}^{p-1})^F$.

a) Consider the first case. If the p-th Betti number of N vanishes, we get a contradiction with Proposition 4.12; if, on the contrary, the p-th Betti number of N is different from zero, Proposition 4.12 implies that

$$\mu \geq \left(\frac{n-2(p+1)+1}{2}\right)^2 b^2 = \left(\frac{n-2p-1}{2}\right)^2 b^2.$$

b) Consider the second case. If the $(p-1)$-th Betti number of N vanishes, we get a contradiction with Proposition 4.11; if, on the contrary, the $(p-1)$-th Betti number of N is different from zero, by Proposition 4.12 we have that

$$\mu \geq \left(\frac{n-2(p-1)-1}{2}\right)^2 b^2 = \left(\frac{n-2p+1}{2}\right)^2 b^2.$$

Combining these facts we complete the proof. □

As in the proof of Lemma 4.6, it is essential that $\mu > 0$. As a consequence, all we can say about the essential spectrum of $(\Delta_{M3}^p)^F$ is:

Proposition 4.16. *Let $a = -1$, $b > 0$, $1 \leq p \leq n-1$.*

(1) *If both the p-th Betti number and the $(p-1)$-th Betti number of N vanish, $\sigma_{\text{ess}}((\Delta_{M3}^p)^F) \setminus \{0\} = \emptyset$.*

(2) *If the p-th Betti number of N is different from zero and the $(p-1)$-th Betti number of N vanishes,*

$$\sigma_{\text{ess}}((\Delta_{M3}^p)^F) \setminus \{0\} \subseteq \left[\left(\frac{n-2p-1}{2}\right)^2 b^2, +\infty\right).$$

(3) *If the p-th Betti number of N vanishes and the $(p-1)$-th Betti number of N is different from zero,*

$$\sigma_{\text{ess}}((\Delta_{M3}^p)^F) \setminus \{0\} \subseteq \left[\left(\frac{n-2p+1}{2}\right)^2 b^2, +\infty\right).$$

(4) *If both the p-th and the $(p-1)$-th Betti numbers of N are different from zero,*

$$\sigma_{\text{ess}}((\Delta^p_{M3})^F) \setminus \{0\} \subseteq$$

$$\subseteq \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right).$$

Combining Proposition 4.11, Proposition 4.12 and Proposition 4.16, we finally get the following Theorem, which shows an interesting link between the cohomology of the boundary N and the essential spectrum of Δ^p_M (with 0 excluded):

Theorem 4.17. *Let $a = -1$, $b > 0$, $0 \leq p \leq n$.*

(1) *If both the p-th and the $(p-1)$-th Betti numbers of N vanish,*

$$\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\} = \emptyset.$$

(2) *If the p-th Betti number of N is different from zero and the $(p-1)$-th Betti number of N vanishes,*

$$\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\} = \left[\left(\frac{n-2p-1}{2}\right)^2 b^2, +\infty\right).$$

(3) *If the p-th Betti number of N vanishes and the $(p-1)$-th Betti number of N is different from zero,*

$$\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\} = \left[\left(\frac{n-2p+1}{2}\right)^2 b^2, +\infty\right).$$

(4) *If both the p-th and the $(p-1)$-th Betti numbers of N are different from zero,*

$$\sigma_{\text{ess}}(\Delta^p_M) \setminus \{0\} =$$

$$= \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right).$$

5. The case $a<-1$

Let us perform the change of variables

$$(0,+\infty)\longrightarrow\left(\frac{1}{|a+1|},+\infty\right)$$

$$r(t):=\frac{e^{-(a+1)t}}{|a+1|}.$$

The Riemannian metric in the new coordinate system (r,θ) on $(\tilde{c},+\infty)\times N$, where $\tilde{c}=r(c)$, takes the form

$$ds'^2=dr^2+|a+1|^{\frac{2b}{a+1}}\,r^{\frac{2b}{a+1}}\,d\theta_N^2.$$

If we apply the decomposition techniques in the new coordinate system, we find the operators $D_{1\lambda_k^p}$, $D_{2\lambda_k^{p-1}}$, $D_{3\lambda_k^{p-1}}$, defined on the smooth functions with compact support in $(\tilde{c},+\infty)$:

$$D_{1\lambda_k^p}w=-\frac{\partial^2 w}{\partial r^2}+K_1(n,p)r^{-2}w+\lambda_k^p|a+1|^{\frac{-2b}{a+1}}\,r^{\frac{-2b}{a+1}}w,$$

$$D_{2\lambda_k^{p-1}}w=-\frac{\partial^2 w}{\partial r^2}+K_2(n,p)r^{-2}w+\lambda_k^{p-1}|a+1|^{\frac{-2b}{a+1}}\,r^{\frac{-2b}{a+1}}w,$$

$$\begin{aligned}D_{3\lambda_k^{p-1}}(w_1\oplus w_2)=&\left(-\frac{\partial^2 w_1}{\partial r^2}+K_1(n,p)r^{-2}w_1+\lambda_k^{p-1}|a+1|^{\frac{-2b}{a+1}}r^{\frac{-2b}{a+1}}w_1+\right.\\&\left.+\sqrt{\lambda_k^{p-1}}|a+1|^{-\frac{b}{a+1}}r^{-\frac{b}{a+1}-1}w_2\right)\oplus\\&\oplus\left(-\frac{\partial^2 w_2}{\partial r^2}+K_2(n,p)r^{-2}w_2+\lambda_k^{p}|a+1|^{\frac{-2b}{a+1}}r^{\frac{-2b}{a+1}}w_2+\right.\\&\left.+\sqrt{\lambda_k^{p-1}}|a+1|^{-\frac{b}{a+1}}r^{-\frac{b}{a+1}-1}w_1\right),\end{aligned}$$

where

$$K_1(n,p)=\left(\frac{n-2p-1}{2}\right)^2\frac{b^2}{(a+1)^2}+\frac{n-2p-1}{2}\frac{b}{|a+1|},$$

$$K_2(n,p)=\left(\frac{n-2p+1}{2}\right)^2\frac{b^2}{(a+1)^2}+\frac{n-2p+1}{2}\frac{b}{|a+1|}.$$

Since the potential terms containing p and n tend to zero as $r\to+\infty$, we can presume that for $a=-1$ the bottom of the essential spectrum of Δ_M^p will not depend on the relationships between the dimension n of M and the degree p. The asymptotic behaviour of the potential is again strongly determined by the sign of b. Hence also for $a<-1$ we will consider separately the cases $b<0$, $b=0$, $b>0$.

5.1. **The case $b < 0$.** For $a < -1$, $b < 0$, all the potential terms in the operators $D_{1\lambda_k^p}$, $D_{2\lambda_k^{p-1}}$, $D_{3\lambda_k^{p-1}}$ tend to zero as $r \to +\infty$; hence the following result is not surprising:

Theorem 5.1. *Let $a < -1$, $b < 0$. Then, for every $p \in [0, n]$,*

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = [0, +\infty).$$

Proof. Since $\sigma(\Delta_M^p) \subseteq [0, +\infty]$, it suffices to show that, for every $k \in \mathbb{N}$, $\sigma_{\mathrm{ess}}(D_{1\lambda_k^p}) = [0, +\infty)$.

To this purpose, let us consider the Friedrichs extension $(D_{10})^F$ of the Laplacian $-\frac{\partial^2}{\partial r^2}$ on $C_c^\infty(\tilde{c}, +\infty)$. We will show that $(D_{1\lambda_k^p})^F - (D_{10})^F$ is a relatively compact perturbation of $(D_{10})^F$.

First of all, since $X_{D_{1\lambda_k^p}} = X_{D_{10}}$ and $\mathcal{D}(D^*_{1\lambda_k^p}) = \mathcal{D}(D^*_{10})$, we have that $\mathcal{D}((D_{10})^F) \subseteq \mathcal{D}((D_{1\lambda_k^p})^F - (D_{10})^F) = \mathcal{D}((D_{10})^F)$.

Now, $(D_{1\lambda_k^p})^F - (D_{10})^F$ is given by

$$(D^F_{1\lambda_k^p} - D^F_{10})w = (K_1(n,p)r^{-2} + \lambda_k^p|a+1|^{-\frac{2b}{a+1}}\, r^{-\frac{2|b|}{|a+1|}})\, w,$$

where

$$K_1(n,p)r^{-2} \in L^2(\tilde{c}, +\infty) \cap L^\infty(\tilde{c}, +\infty)$$

and

$$\lambda_k^p\, |a+1|^{-\frac{2|b|}{|a+1|}} r^{-\frac{2|b|}{|a+1|}} \in L^\infty_\varepsilon(\tilde{c}, +\infty)$$

for $0 < \varepsilon \leq \frac{|b|}{|a+1|}$ since

$$(1 + r^2)^\varepsilon r^{-\frac{2|b|}{|a+1|}} \leq 2^\varepsilon r^{2(\varepsilon - \frac{|b|}{|a+1|})}.$$

If $\{w_n\} \subset \mathcal{D}((D_{01})^F)$ is such that

$$\|w_n\|_{L^2(\tilde{c},+\infty)} + \|(D_{01})^F w_n\|_{L^2(\tilde{c},+\infty)} \leq C,$$

then

$$\|w_n\|^2_{L^2(\tilde{c},+\infty)} + \|\frac{\partial w_n}{\partial r}\|^2_{L^2(\tilde{c},+\infty)} \leq C,$$

hence $\{w_n\}$ is a bounded sequence in $W^{1,2}(\tilde{c}, +\infty)$, in $L^\infty(\tilde{c}, +\infty)$ and in $W^{1,2}(K)$ for every compact subset $K \subset (\tilde{c}, +\infty)$.

We have

$$\|((D_{1\lambda_k^p})^F - (D_{10})^F)(w_n - w_m)\|_{L^2(\tilde{c},+\infty)} \leq$$

$$\leq C_1\|r^{-2}(w_n - w_m)\|_{L^2(\tilde{c},+\infty)} + \lambda_k^p C_2 \|r^{-\frac{2|b|}{|a+1|}}(w_n - w_m)\|_{L^2(\tilde{c},+\infty)}.$$

As for the first term, since $r^{-2} \in L^2(\tilde{c}, +\infty) \cap L^\infty(\tilde{c}, +\infty)$, following the same argument as in Lemma 4.1 we find a subsequence, again denoted by $\{w_n\}$, such that for every $\eta > 0$ there exists $\bar{n}$ for which

$$n, m > \bar{n} \quad \Longrightarrow C_1\|r^{-2}(w_n - w_m)\|_{L^2(\tilde{c},+\infty)} < \frac{\eta}{2}.$$

As for the second term, we cannot apply the same argument because $r^{-\frac{2|b|}{|a+1|}}$ might not belong to $L^2(\tilde{c},+\infty)$. Nevertheless, since $r^{-\frac{2|b|}{|a+1|}} \leq C(1+r^2)^{-\varepsilon}$ for $0<\varepsilon<\frac{|b|}{|a+1|}$, we have that, for every $d>\tilde{c}$,

$$\int_{\tilde{c}}^{+\infty} x^{-\frac{4|b|}{|a+1|}}(w_n-w_m)^2\,dx \leq C\int_{\tilde{c}}^{+\infty}(1+x^2)^{-2\varepsilon}(w_n-w_m^2)^2\,dx =$$

$$= C\int_{\tilde{c}}^{d}(1+x^2)^{-2\varepsilon}(w_n-w_m)^2\,dx + C\int_{d}^{+\infty}(1+x^2)^{-2\varepsilon}(w_n-w_m)^2\,dx \leq$$

$$\leq C\|w_n-w_m\|^2_{L^2(\tilde{c},d)} + C\|w_n-w_m\|_{L^\infty(\tilde{c},+\infty)}\frac{1}{(1+d^2)^{2\varepsilon}} \leq$$

$$\leq C\|w_n-w_m\|^2_{L^2(\tilde{c},d)} + \tilde{C}\frac{1}{(1+d^2)^{2\varepsilon}}.$$

for some positive constant $\tilde{C}$. Let $\{\tilde{c}_h\}\subset(\tilde{c},+\infty)$ be a sequence such that $\tilde{c}_h\to+\infty$ as $h\to+\infty$ and, for every $h\in\mathbb{N}$,

$$\frac{\tilde{C}}{(1+\tilde{c}_h^2)^{2\varepsilon}} < \frac{1}{h}.$$

Then, again through an argument similar to that of Lemma 4.1, we can extract a subsequence, again denoted by $\{w_n\}$ for shortness, such that, for every $\eta>0$ and for every $h\in\mathbb{N}$, there exists $\bar{n}(h)$ for which

$$n,m>\bar{n}(h) \quad\Longrightarrow\quad \int_{\tilde{c}}^{+\infty} x^{-\frac{4|b|}{|a+1|}}(w_n-w_m)^2\,dx < \frac{\eta}{2}+\frac{1}{h}.$$

Hence, we have found a subsequence $\{w_n\}$ such that for every $\eta>0$ and for every $h>0$, there exists $\tilde{n}=\max\{\bar{n},\bar{n}(h)\}$ for which

$$n,m>\tilde{n} \quad\Longrightarrow\quad \|((D_{1\lambda_k^p})^F-(D_{10})^F)(w_n-w_m)\|_{L^2(\tilde{c},+\infty)} \leq \eta+\frac{1}{h}.$$

As a consequence, $(D_{1\lambda_k^p})^F-(D_{10})^F$ is a relatively compact perturbation of $(D_{10})^F$. □

5.2. **The case $b=0$.** For $a<-1$, $b=0$, the operators $D_{1\lambda_k^p}$, $D_{2\lambda_k^{p-1}}$, $D_{3\lambda_k^{p-1}}$ are simply

$$D_{1\lambda_k^p}w = -\frac{\partial^2 w}{\partial r^2}+\lambda_k^p w,$$

$$D_{2\lambda_k^{p-1}}w = -\frac{\partial^2 w}{\partial r^2}+\lambda_k^{p-1} w,$$

$$D_{3\lambda_k^{p-1}}(w_1\oplus w_2) = \left(-\frac{\partial^2 w_1}{\partial r^2}+\lambda_k^{p-1}w_1\right)\oplus\left(-\frac{\partial^2 w_2}{\partial r^2}+\lambda_k^{p-1}w_2\right).$$

Hence, arguing as in the case $a=-1$, $b=0$ we get the following result:

Theorem 5.2. *Let $a < -1$, $b = 0$. Then, for $0 \leq p \leq n$, the essential spectrum of Δ^p_M is given by*

$$\sigma_{\mathrm{ess}}(\Delta^p_M) = \bigcup_k ([\lambda^p_k, +\infty) \cup [\lambda^{p-1}_k, +\infty)) = [\overline{\lambda}, +\infty),$$

where $\overline{\lambda} = \min_k \left\{\lambda^p_k, \lambda^{p-1}_k\right\}$.

In particular, if the p-th or the $(p-1)$-th Betti number of N does not vanish, $\overline{\lambda} = 0$. Otherwise, $\overline{\lambda} > 0$.

5.3. **The case $b > 0$.** Once more, we begin with the spectral analysis of $(D_{1\lambda^p_k})^F$ for every $k \in \mathbb{N}$:

Lemma 5.3. *Let $a < -1$, $b > 0$. For any $k \in \mathbb{N}$, if $\lambda^p_k = 0$, $\sigma_{\mathrm{ess}}((D_{1\lambda^p_k})^F) = [0, +\infty)$; if, on the contrary, $\lambda^p_k > 0$, $\sigma_{\mathrm{ess}}((D_{1\lambda^p_k})^F) = \emptyset$.*

Proof. If $\lambda^p_k = 0$,

$$D_{1\lambda^p_k} = -\frac{\partial^2}{\partial r^2} + K_1(n,p) r^{-2},$$

and since

$$r^{-2} \in L^2(\tilde{c}, +\infty) \cap L^\infty(\tilde{c}, +\infty),$$

the essential spectrum of $(D_{1\lambda^p_k})^F$ coincides with the essential spectrum of the Friedrichs extension of the Laplacian $-\frac{\partial^2}{\partial r^2}$ on $C^\infty_c(\tilde{c}, +\infty)$. Hence, if $\lambda^p_k = 0$, $\sigma_{\mathrm{ess}}((D_{1\lambda^p_k})^F) = [0, +\infty)$.

If, on the contrary, $\lambda^p_k > 0$, for every $h \in (0,1)$,

$$\lim_{r \to +\infty} \int_r^{r+h} \left(K_1(n,p) s^{-2} + \lambda^p_k |a+1|^{\frac{2b}{|a+1}} s^{-\frac{2b}{a+1}} \right) ds =$$

$$= \lim_{r \to +\infty} \lambda^p_k \left(\frac{4b}{|a+1|} + 1 \right)^{-1} \left[(r+h)^{\frac{2b}{|a+1|}+1} - r^{\frac{2b}{|a+1|}+1} \right] = +\infty$$

since $\frac{2b}{|a+1|} > 0$. Then, in view of Theorem 3.13 in [9], $\sigma_{\mathrm{ess}}(D_{1\lambda^p_k}) = \emptyset$. $\square$

If $K_1(n,p) \geq 0$, we find

$$\langle (D_{1\lambda^p_k})^F w, w \rangle_{L^2(\tilde{c}, +\infty)} \geq \int_{\tilde{c}}^{+\infty} \lambda^p_k |a+1|^{\frac{2b}{|a+1|}} s^{\frac{2b}{|a+1|}} w^2 \, ds \geq$$

$$\geq C |a+1|^{\frac{2b}{|a+1|}} \lambda^p_k \langle w, w \rangle_{L^2(\tilde{c}, +\infty)}.$$

If $K_1(n,p) < 0$, since, again, $\tilde{c} > 0$ can be chosen arbitrarily large, we can suppose that for every $r > \tilde{c}$

$$V(r) := K_1(n,p) r^{-2} + \lambda^p_k |a+1|^{\frac{2b}{|a+1}} r^{\frac{2b}{|a+1|}} > V(\tilde{c}) > 0;$$

hence,

$$\langle (D_{1\lambda_k^p})^F w, w\rangle_{L^2(\tilde{c},+\infty)} \geq V(\tilde{c})\langle w, w\rangle_{L^2(\tilde{c},+\infty)} = \tag{5.1}$$
$$= \left(\frac{K_1(n,p)}{\tilde{c}^2} + \lambda_k^p|a+1|^{\frac{2b}{|a+1|}}\tilde{c}^{\frac{2b}{|a+1|}}\right)\langle w, w\rangle_{L^2(\tilde{c},+\infty)}.$$

In both cases, for any w such that $\|w\|_{L^2(\tilde{c},+\infty)} = 1$

$$\langle (D_{1\lambda_k^p})^F w, w\rangle_{L^2(\tilde{c},+\infty)} \to +\infty$$

as $k \to +\infty$. As a consequence:

Proposition 5.4. *Let $a < -1$, $b > 0$. For $0 \leq p \leq (n-1)$, if the p-th Betti number of N vanishes, then $\sigma_{\mathrm{ess}}((\Delta^p_{M1})^F) = \emptyset$. If, on the contrary, the p-th Betti number of N is different from zero, then $\sigma_{\mathrm{ess}}((\Delta^p_{M1})^F) = [0,+\infty)$.*

By duality,

Proposition 5.5. *Let $a < -1$, $b > 0$. For $1 \leq p \leq n$, if the $(p-1)$-th Betti number of N vanishes, then $\sigma_{\mathrm{ess}}((\Delta^p_{M2})^F) = \emptyset$; if, on the contrary, the $(p-1)$-th Betti number of N is different from zero, then $\sigma_{\mathrm{ess}}((\Delta^p_{M2})^F) = [0,+\infty)$.*

We still have to investigate the essential spectrum of $(\Delta^p_{M3})^F$. First of all, we will compute the essential spectrum of $D_{3\lambda_k^{p-1}}$ for every $k \in \mathbb{N}$. In analogy with what we did in the case $a = -1$, $b > 0$, we need a preliminary Lemma:

Lemma 5.6. *For every $K \in \mathbb{R}$, the essential spectrum of the Friedrichs extension D^F of the operator*

$$D : C_c^\infty(c,+\infty) \oplus C_c^\infty(c,+\infty) \longrightarrow L^2(c,+\infty) \oplus L^2(c,+\infty)$$

defined by

$$D(w_1 \oplus w_2) = \left(-\frac{\partial^2 w_1}{\partial r^2} + Kr^{-2}w_1 + \lambda_k^{p-1}|a+1|^{-\frac{2b}{a+1}}r^{-\frac{2b}{a+1}}w_1 + \right.$$
$$\left. + \sqrt{\lambda_k^{p-1}}\,|a+1|^{-\frac{b}{a+1}}r^{-\frac{b}{a+1}-1}w_2\right) \oplus$$
$$\oplus \left(-\frac{\partial^2 w_2}{\partial r^2} + Kr^{-2}w_2 + \lambda_k^{p-1}|a+1|^{-\frac{2b}{a+1}}r^{-\frac{2b}{a+1}}w_2 + \right.$$
$$\left. + \sqrt{\lambda_k^{p-1}}|a+1|^{-\frac{b}{a+1}}r^{-\frac{b}{a+1}-1}\right)$$

is empty.

Proof. First of all, since as previously stated the essential spectrum of D^F does not depend on the choice of the first endpoint $\tilde{c}$ of $(\tilde{c}, +\infty)$, given K and $\mu > 0$ we can suppose that for some positive constant C, for every $t > \tilde{c}$

$$Kr^{-2} + \lambda_k^{p-1}|a+1|^{2\frac{|b|}{|a+1|}} r^{2\frac{|b|}{|a+1|}} \pm \sqrt{\lambda_k^{p-1}}|a+1|^{\frac{|b|}{|a+1|}} r^{\frac{|b|}{|a+1|}-1} - \mu \geq C. \tag{5.2}$$

As in the proof of Lemma 4.13, let us consider the orthogonal decomposition (4.6)

$$L^2(\tilde{c}, +\infty) \oplus L^2(\tilde{c}, +\infty) = \mathcal{V}_1 \oplus \mathcal{V}_2,$$

where

$$\mathcal{V}_1 := \{w_1 \oplus w_2 \,|\, w_1 = w_2\}$$

and

$$\mathcal{V}_2 := \{w_1 \oplus w_2 \,|\, w_1 = -w_2\}\,.$$

D is invariant under (4.6), hence

$$D^F = (D_{|\mathcal{V}_1})^F \oplus (D_{|\mathcal{V}_2})^F,$$

and

$$\sigma_{\mathrm{ess}}(D^F) = \sigma_{\mathrm{ess}}(D^F_{|\mathcal{V}_1}) \cup \sigma_{\mathrm{ess}}(D^F_{|\mathcal{V}_2}).$$

If $\mu \in \sigma_{\mathrm{ess}}(D^F_{|\mathcal{V}_1})$, there exists a Weyl sequence $\{w_n\} \subset \mathcal{D}(D_{|\mathcal{V}_1^F})$ for μ: we have that $\|w_n\|_{L^2(\tilde{c},+\infty)} \leq C$, $(D-\mu)(w_n \oplus w_n) \longrightarrow 0$ as $n \to +\infty$, but $\{w_n\}$ has no convergent subsequence. Then

$$\langle (D-\mu)(w_n \oplus w_n), (w_n \oplus w_n)\rangle_{L^2(\tilde{c},+\infty)\oplus L^2(\tilde{c},+\infty)} \longrightarrow 0$$

as $n \to +\infty$; hence

$$\int_{\tilde{c}}^{+\infty} \left(\frac{\partial w_n}{\partial x}\right)^2 dx + \Big(Kx^{-2} + \lambda_k^{p-1}|a+1|^{2\frac{|b|}{|a+1|}} x^{2\frac{|b|}{|a+1|}} + \\ - \sqrt{\lambda_k^{p-1}}|a+1|^{\frac{|b|}{|a+1|}} x^{\frac{|b|}{|a+1|}-1} - \mu\Big) w_n^2\, dx \longrightarrow 0 \tag{5.3}$$

as $n \to +\infty$. Since the estimate (5.2) implies that $\|w_n\|_{L^2(\tilde{c},+\infty)} \to 0$ as $n \to +\infty$, we obtain that $\mu \notin \sigma_{\mathrm{ess}}(D^F_{|\mathcal{V}_1})$.

Suppose now that $\mu \in \sigma_{\mathrm{ess}}(D^F_{|\mathcal{V}_2})$. If $\{w_n\}$ is a Weyl sequence for μ for $D^F_{|\mathcal{V}_2}$, we have

$$\langle (D-\mu)(w_n \oplus -w_n), (w_n \oplus -w_n)\rangle_{L^2(\tilde{c},+\infty)\oplus L^2(\tilde{c},+\infty)} \longrightarrow 0$$

as $n \to +\infty$. Again in view of the estimate (5.2), $\|w_n\|_{L^2(\tilde{c},+\infty)} \to 0$ as $n \to +\infty$. But a Weyl sequence cannot converge. Hence, $\mu \notin \sigma_{\mathrm{ess}}(D^F_{|\mathcal{V}_2})$. This completes the proof. $\square$

We can now compute the essential spectrum of $(D_{3\lambda_k^{p-1}})^F$ for every $k \in \mathbb{N}$:

Lemma 5.7. *For $1 \leq p \leq n-1$ and for every $k \in \mathbb{N}$,*

$$\sigma_{\text{ess}}((D_{3\lambda_k^{p-1}})^F) = \emptyset.$$

Proof. Let us consider the Friedrichs extension $(D_{30})^F$ of the operator

$$D_{30} : C_c^\infty(\tilde{c}, +\infty) \oplus C_c^\infty(\tilde{c}, +\infty) \longrightarrow L^2(\tilde{c}, +\infty) \oplus L^2(\tilde{c}, +\infty)$$

$$\begin{aligned} D_{30}(w_1 \oplus w_2) = \Bigg(&-\frac{\partial^2 w_1}{\partial r^2} + \bar{K} r^{-2} w_1 + \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} r^{-\frac{2b}{a+1}} w_1 + \\ &+ \sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} r^{-\frac{b}{a+1}-1} w_2 \Bigg) \oplus \\ \oplus \Bigg(&-\frac{\partial^2 w_2}{\partial r^2} + \bar{K} r^{-2} w_2 + \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} r^{-\frac{2b}{a+1}} w_2 + \\ &+ \sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} r^{-\frac{b}{a+1}-1} w_1 \Bigg), \end{aligned}$$

where

$$\bar{K} = \max\{K_1(n,p), K_2(n,p)\}.$$

From the previous Lemma, we know that $\sigma_{\text{ess}}((D_{30})^F) = \emptyset$. We will show that $((D_{3\lambda_k^{p-1}})^F - (D_{30})^F)$ is a relatively compact perturbation of $(D_{30})^F$. First of all, a straightforward computation shows that $X_{D_{30}} \subseteq X_{D_{3\lambda_k^{p-1}}}$ and $\mathcal{D}(D_{30}^*) \subseteq \mathcal{D}(D^*_{3\lambda_k^{p-1}})$; hence, $\mathcal{D}((D_{30})^F) \subseteq \mathcal{D}((D^F_{3\lambda_k^{p-1}}))$, whence $\mathcal{D}((D_{3\lambda_k^{p-1}})^F - (D_{30})^F) = \mathcal{D}((D_{30})^F)$.

We still have to show that, given a sequence $\{w_{1n} \oplus w_{2n}\} \subset \mathcal{D}((D_{30})^F)$ such that

$$\|w_{1n} \oplus w_{2n}\|^2_{L^2 \oplus L^2} + \|((D_{30})^F)(w_{1n} \oplus w_{2n})\|^2_{L^2 \oplus L^2} \leq C,$$

there exists a subsequence $\{w_{1n_k} \oplus w_{2n_k}\}$ such that

$$\left\{((D_{3\lambda_k^{p-1}})^F - (D_{30})^F)(w_{1n_k} \oplus w_{2n_k})\right\}$$

converges.

Now, the fact that $\|(D_{30})^F(w_{1n} \oplus w_{2n})\|_{L^2(\tilde{c},+\infty) \oplus L^2(\tilde{c},+\infty)} \leq C$ is equivalent to the inequalities

$$\begin{aligned} (5.4) \quad \| &-\frac{\partial^2 w_{1n}}{\partial r^2} + \bar{K} r^{-2} w_{1n} + \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} r^{-\frac{2b}{a+1}} w_{1n} + \\ &+ \sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} r^{-\frac{b}{a+1}-1} w_{2n} \|_{L^2(\tilde{c},+\infty)} \leq C, \end{aligned}$$

$$\| - \frac{\partial^2 w_{2n}}{\partial r^2} + \bar{K} r^{-2} w_{2n} + \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} r^{-\frac{2b}{a+1}} w_{2n} + \\ + \sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} r^{-\frac{b}{a+1}-1} w_{1n} \|_{L^2(\tilde{c},+\infty)} \leq C, \tag{5.5}$$

which in turn imply

$$\| - \frac{\partial^2 (w_{1n}+w_{2n})}{\partial r^2} + \bar{K} r^{-2} (w_{1n}+w_{2n}) + \\ + \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} r^{-\frac{2b}{a+1}} (w_{1n}+w_{2n}) + \\ + \sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} r^{-\frac{b}{a+1}-1} (w_{1n}+w_{2n}) \|_{L^2(\tilde{c},+\infty)} \leq C, \tag{5.6}$$

$$\| - \frac{\partial^2 (w_{1n}-w_{2n})}{\partial r^2} + \bar{K} r^{-2} (w_{1n}-w_{2n}) + \\ + \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} r^{-\frac{2b}{a+1}} (w_{1n}-w_{2n}) + \\ + \sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} r^{-\frac{b}{a+1}-1} (w_{2n}-w_{1n}) \|_{L^2(\tilde{c},+\infty)} \leq C. \tag{5.7}$$

By multiplication with $(w_{1n}+w_{2n})$ and $(w_{1n}-w_{2n})$ respectively, (5.6) and (5.7) yield:

$$\int_{\tilde{c}}^{+\infty} \left(\frac{\partial (w_{1n}+w_{2n})}{\partial s} \right)^2 ds + \int_{\tilde{c}}^{+\infty} \bar{K} s^{-2} (w_{1n}+w_{2n})^2 \, ds \\ + \int_{\tilde{c}}^{+\infty} \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} s^{-\frac{2b}{a+1}} (w_{1n}+w_{2n})^2 \, ds + \\ + \int_{\tilde{c}}^{+\infty} \sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} s^{-\frac{b}{a+1}-1} (w_{2n}+w_{1n})^2 \, ds \leq C, \tag{5.8}$$

$$\int_{\tilde{c}}^{+\infty} \left(\frac{\partial (w_{1n}-w_{2n})}{\partial s} \right)^2 ds + \int_{\tilde{c}}^{+\infty} \bar{K} s^{-2} (w_{1n}-w_{2n})^2 \, ds + \\ + \int_{\tilde{c}}^{+\infty} \lambda_k^{p-1} |a+1|^{-\frac{2b}{a+1}} s^{-\frac{2b}{a+1}} (w_{1n}-w_{2n})^2 \, ds + \\ + \int_{/tildec}^{+\infty} -\sqrt{\lambda_k^{p-1}} |a+1|^{-\frac{b}{a+1}} s^{-\frac{b}{a+1}-1} (w_{1n}-w_{2n})^2 \, ds \leq C. \tag{5.9}$$

Now, there exists $\bar{c}$ such that for every $k \in \mathbb{N}$ and for every $r > \bar{c}$

$$\bar{K} r^{-2} + \lambda_k^{p-1} |a+1|^{\frac{|2b|}{|a+1|}} r^{\frac{|2b|}{|a+1|}} \pm \sqrt{\lambda_k^{p-1}} |a+1|^{\frac{|2b|}{|a+1|}} r^{\frac{|b|}{|a+1|}-1} \geq C$$

for some positive constant C. Since, in view of the results of [5], the essential spectrum of $(D_{3\lambda_k^{p-1}})^F$ does not depend on the first endpoint

$\tilde{c}$ of $(\tilde{c}, +\infty)$, we can suppose that $\tilde{c} > \bar{c}$; then, from the estimates (5.8) and (5.9) we obtain that

$$\int_{\tilde{c}}^{+\infty} (w_{1n} \pm w_{2n})^2\, ds \leq C.$$

Hence, $\{w_{1n}\}$ and $\{w_{2n}\}$ are bounded in $W^{1,2}(\tilde{c}, +\infty)$; as a consequence, they are bounded also in $L^\infty(\tilde{c}, +\infty)$ and in $W^{1,2}(K)$ for every compact subset $K \subset (\tilde{c}, +\infty)$. Now, for every $n, m \in \mathbb{N}$

$$\|((D_{3\lambda_k^{p-1}})^F - (D_{30})^F)((w_{1n} - w_{1m}) \oplus (w_{2n} - w_{2m}))\|^2_{L^2(\tilde{c},+\infty)\oplus L^2(\tilde{c},+\infty)} =$$

$$= \sum_{i=1,2} \int_{\tilde{c}}^{+\infty} (K_i(p,n) - \bar{K})^2 s^{-4} (w_{in} - w_{im})^2\, ds;$$

since $s^{-2} \in L^2(\tilde{c}, +\infty) \cap L^\infty(\tilde{c}, +\infty)$, following the argument of Lemma 4.1 the conclusion follows. $\square$

Moreover,

Lemma 5.8. *Let $a < -1$, $b > 0$. If the p-th and the $(p-1)$-th Betti numbers of N both vanish, and if $0 < \mu \in \overline{\sigma_p((\Delta^p_{M3})^F)}$, then μ is an isolated eigenvalue of finite multiplicity.*

Proof. If μ is an eigenvalue of infinite multiplicity of $(\Delta^p_{M3})^F$ or is a cluster point of $\sigma_p((\Delta^p_{M3})^F)$, there exist a sequence $\{\mu_k\}$ of eigenvalues of $(\Delta^p_{M3})^F$ and a corresponding sequence of normalized, mutually orthogonal eigenforms $\{\Phi_k\}$ such that for every k

$$(\Delta^p_{M3})^F \Phi_k - \mu_k \Phi_k = 0$$

and

$$\mu_k \longrightarrow \mu \qquad \text{as } k \to +\infty.$$

In view of the weak Kodaira decomposition, replacing $\{\Phi_k\}$ by a subsequence (again denoted, for shortness, by the same symbol) we can suppose that either $\delta^p_M \Phi_k = 0$ for every $k \in \mathbb{N}$ or $d^p_M \Phi_k = 0$ for every $k \in \mathbb{N}$. Following the argument of Lemma 4.6, we find that in the first case μ lies in the essential spectrum of the operator $(\Delta^{p+1}_{M2})^F$, whilst in the second case μ lies in the essential spectrum of the operator $(\Delta^{p-1}_{M1})^F$. In the first case, since by assumption the p-th Betti number of N vanishes, we get a contradiction with Proposition 5.5. In the second case, since by assumption the $(p-1)$-th Betti number of N vanishes, we get a contradiction with Proposition 5.4. $\square$

Summing up, for $a < -1$, $b > 0$ we have the following result:

Theorem 5.9. *Let $a < -1$, $b > 0$, $p \in [0, n]$. If both the p-th and the $(p-1)$-th Betti numbers of N vanish, then $\sigma_{\mathrm{ess}}(\Delta_M^p) \setminus \{0\} = \emptyset$. If, on the contrary, at least one of them is different from zero, then $\sigma_{\mathrm{ess}}(\Delta_M^p) = [0, +\infty)$.*

6. The rotationally symmetric case: $N = \mathbb{S}^{n-1}$

Since the techniques in Lemma 4.6, 4.15, 5.8 work only when $\mu_k > 0$, in the case of a general manifold with boundary $\overline{M}$ for some values of the parameters a and b (namely for $a = -1$, $b < 0$, for $a = -1$, $b > 0$ and for $a < -1$, $b > 0$) we were able to compute only $\sigma_{\mathrm{ess}}(\Delta_M^p) \setminus \{0\}$. In particular, in those cases we were not able to establish whether 0 is an eigenvalue of infinite multiplicity of Δ_M^p or not.

In the rotationally symmetric case, that is, when $\overline{M}$ is the unitary ball $\overline{B(\underline{0}, 1)}$ in $\mathbb{R}^n$ and the Riemannian metric is globally invariant under rotations, we can be slightly more precise. In fact, we have the following generalization of a result of Dodziuk (see [3]), whose proof essentially follows that of [3] and is therefore omitted.

Theorem 6.1. *Let us consider, for $n \geq 2$, the manifold*

$$\tilde{M} = [0, +\infty) \times \mathbb{S}^{n-1}$$

endowed with a complete Riemannian metric of the type

$$f(t)\, dt^2 + g(t)\, d\theta^2_{\mathbb{S}^{n-1}}, \tag{6.1}$$

where $f(t) > 0$, $g(t) > 0$ for every $t \in [0, +\infty)$, and $d\theta^2_{\mathbb{S}^{n-1}}$ is the standard metric on $\mathbb{S}^{n-1}$. Let us suppose that for some $\epsilon > 0$

$$f(t) \equiv 1, \quad g(t) = t^2 \quad \text{for } t \in (0, \epsilon). \tag{6.2}$$

If we denote by $\mathcal{H}^p(\tilde{M})$, for $p = 0, ..., n$, the space of L^2 harmonic p-forms on $\tilde{M}$, we have:

1. *for $p \neq 0, n, n/2$, $\mathcal{H}^p(\tilde{M}) = \{0\}$;*
2. *if $\int_0^\infty f^{\frac{1}{2}}(s) g^{\frac{n-1}{2}}(s)\, ds = +\infty$, then $\mathcal{H}^n(\tilde{M}) \simeq \mathcal{H}^0(\tilde{M}) = \{0\}$; if, on the contrary, $\int_0^\infty f^{\frac{1}{2}}(s) g^{\frac{n-1}{2}}(s)\, ds < +\infty$, then $\mathcal{H}^n(\tilde{M}) \simeq \mathcal{H}^0(\tilde{M}) = \mathbb{R}$;*
3. *for $p = \frac{n}{2}$, if $\int_1^{+\infty} f^{\frac{1}{2}}(s) g^{-\frac{1}{2}}(s)\, ds = +\infty$, then $\mathcal{H}^p(\tilde{M}) = \{0\}$; if, on the other hand, $\int_1^{+\infty} f^{\frac{1}{2}}(s) g^{-\frac{1}{2}}(s)\, ds < +\infty$, then $\mathcal{H}^{\frac{n}{2}}(\tilde{M})$ is a Hilbert space of infinite dimension.*

Now, suppose that the metric (6.1) fulfills the asymptotic condition:

$$f(t) = e^{-2(1+a)t}, \quad g(t) = e^{-2bt} \quad \text{for } t > c >> 0. \tag{6.3}$$

A straightforward computation shows that

$$\int_0^{+\infty} f(s)^{\frac{1}{2}} g(s)^{\frac{n-1}{2}}\, ds \begin{cases} = +\infty & \text{if } b \leq -\frac{a+1}{n-1} \\ < +\infty & \text{if } b > -\frac{a+1}{n-1} \end{cases},$$

whilst

$$\int_1^{+\infty} f(s)^{\frac{1}{2}} g(s)^{-\frac{1}{2}}\, ds \begin{cases} = +\infty & \text{if } b \geq a+1 \\ < +\infty & \text{if } b < a+1 \end{cases}.$$

As a consequence, from Theorem 6.1 we can easily deduce the following

Theorem 6.2. *For $n \geq 2$, let us consider the manifold $\tilde{M}$, endowed with a Riemannian metric of type (6.1), satisfying conditions (6.2) and (6.3).*

(1) *If $p \neq n/2, n, 0$, then $0 \notin \sigma_p(\Delta^p_{\tilde{M}})$.*
(2) *If $p = 0, n$, then $0 \in \sigma_p(\Delta^p_{\tilde{M}})$ if and only if $b > -\frac{a+1}{n-1}$.*
(3) *If $p = n/2$, $0 \in \sigma_p(\Delta^p_{\tilde{M}})$ if and only if $b < a+1$, and in this case $\mathcal{H}^p(\tilde{M})$ is a Hilbert space of infinite dimension, hence $0 \in \sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}}) \cap \sigma_p(\Delta^p_{\tilde{M}})$.*

Let us consider, for the moment, the Riemannian manifold $\tilde{M}$. Since the Riemannian metric on $\tilde{M}$ is of type (3.1), the results of sections 4 and 5 can be recovered through the decomposition techniques of section 3. Moreover, since the Riemannian metric is globally invariant under rotations, the decompositions can be applied directly on $\Delta^p_{\tilde{M}}$ and not only on its Friedrichs extension $(\Delta^p_{\tilde{M}})^F$: we can write

$$\Delta^p_{\tilde{M}} = \Delta^p_{\tilde{M}1} \oplus \Delta^p_{\tilde{M}2} \oplus \Delta^p_{\tilde{M}3},$$

where, for $i = 1, 2, 3$, $\Delta^p_{\tilde{M}i}$ is the restriction of $\Delta^p_{\tilde{M}}$ to $\mathcal{L}_i(M)$. Moreover, we have

$$\sigma_{\mathrm{ess}}(\Delta^p_M) = \bigcup_{i=1,2,3} \sigma_{\mathrm{ess}}(\Delta^p_{Mi}),$$

$$\sigma_p(\Delta^M_p) = \sigma_p(\Delta^p_{Mi}).$$

In the proof of Proposition 4.6 we used the fact that if $\{\Phi_k\}$ is a sequence of normalized, mutually orthogonal p-eigenforms of $(\Delta^p_{M3})^F$ corresponding to some positive eigenvalues μ_k and $\mu_k \to \mu > 0$ as $k \to +\infty$, then either μ is in the essential spectrum of $(\Delta^{p-1}_{M1})^F$ or μ is in the essential spectrum of $(\Delta^{p+1}_{M2})^F$. In the rotationally symmetric case, this property can be considerably strengthened; namely, the following Proposition holds:

Proposition 6.3. *If Φ is a p-eigenform for $\Delta^p_{\tilde{M}3}$ corresponding to a positive eigenvalue μ, if $d^p_{\tilde{M}}\Phi \neq 0$ (resp. $\delta^p_{\tilde{M}}\Phi \neq 0$), then $d^p_{\tilde{M}}\Phi$ (resp.*

$\delta^p_{\tilde{M}}\Phi$) *is a* $(p+1)$*-eigenform of* $\Delta^{p+1}_{\tilde{M}2}$ *(resp. a* $(p-1)$*-eigenform of* $\Delta^{p-1}_{\tilde{M}1}$*).*

Combining the results of sections 4, 5 with Theorem 6.2 and with Proposition 6.3 we find:

Proposition 6.4. *Let* $\tilde{M}$ *be as in Theorem 6.2: then*

(1) *if* $a=-1$, $b<0$, $0\in\sigma_{\rm ess}(\Delta^p_{\tilde{M}})$ *if and only if* $p\in\left\{\frac{n}{2},\frac{n+1}{2},\frac{n-1}{2}\right\}$*;*
(2) *if* $a=-1$, $b>0$, *for every* $p\in[0,n]$ $0\notin\sigma_{\rm ess}(\Delta^p_{\tilde{M}})$*;*
(3) *if* $a<-1$ *and* $b>0$, $0\in\sigma_{\rm ess}(\Delta^p_{\tilde{M}})$ *if and only if* $p\in\{0,1,n-1,n\}$.

Proof. 1. Let $a=-1$, $b<0$. For $p=\frac{n\pm1}{2}$, $0\in\sigma_{\rm ess}(\Delta^p_{\tilde{M}})$ thanks to Theorem 4.8. On the other hand, if $p=\frac{n}{2}$, 0 is an eigenvalue of infinite multiplicity by Theorem 6.2, hence $0\in\sigma_{\rm ess}(\Delta^p_{\tilde{M}})$. If, on the contrary, $p\notin\left\{\frac{n-1}{2},\frac{n}{2},\frac{n+1}{2}\right\}$, then $0\notin\sigma_p(\Delta^p_{\tilde{M}})$ by Theorem 6.2; hence, in view of Proposition 4.3, if $0\in\sigma_{\rm ess}(\Delta^p_{\tilde{M}})$ there exists a sequence $\{\mu_k\}$ of positive eigenvalues of $\Delta^p_{\tilde{M}3}$ and a corresponding sequence of normalized, mutually orthogonal p-eigenforms $\{\Phi_k\}$ of Δ^p_{M3} such that $\mu_k\to0$ as $k\to+\infty$. Thanks to Proposition 6.3, either $0\in\sigma_{\rm ess}(\Delta^{p-1}_{\tilde{M}1})$ or $0\in\sigma_{\rm ess}(\Delta^{p+1}_{\tilde{M}2})$. Then, in view of Proposition 4.4, we should have

$$0\geq\left(\frac{n-2(p-1)-1}{2}\right)^2b^2=\left(\frac{n-2p+1}{2}\right)^2b^2$$

or

$$0\geq\left(\frac{n-2(p+1)+1}{2}\right)^2b^2=\left(\frac{n-2p-1}{2}\right)^2b^2;$$

since $p\neq\frac{n\pm1}{2}$, we get a contradiction.

2. Let $a=-1$, $b>0$. Since $N=\mathbb{S}^{n-1}$, the p-th Betti number of N is different from zero if and only if $p=0$ or $p=n-1$. As a consequence, in view of Propositions 4.11 and 4.12, we obtain that, for $p\notin\{0,1,n-1,n\}$, $\sigma_{\rm ess}(\Delta^p_{\tilde{M}1})=\sigma_{\rm ess}(\Delta^p_{\tilde{M}2})=\emptyset$. Since Theorem 6.2 implies that for $p\notin\{0,1,n-1,n\}$ 0 cannot be an eigenvalue of $\Delta^p_{\tilde{M}}$, 0 can lie in the essential spectrum of $\Delta^p_{\tilde{M}}$ if and only if there exists a sequence $\{\mu_k\}$ of positive eigenvalues of $\Delta^p_{\tilde{M}3}$ and a corresponding sequence $\{\Phi_k\}$ of normalized, mutually orthogonal p-eigenforms of $\Delta^p_{\tilde{M}3}$ such that $\mu_k\to0$ as $k\to+\infty$. Then, Proposition 6.3 implies that either $0\in\sigma_{\rm ess}(\Delta^{p-1}_{\tilde{M}1})$ or $0\in\sigma_{\rm ess}(\Delta^{p+1}_{\tilde{M}2})$. Since $p-1\notin\{0,n-1\}$ and $(p+1)-1\notin\{0,n-1\}$, we get a contradiction with Proposition 4.11 or with Proposition 4.12. Hence, if $p\notin\{0,1,n-1,n\}$, then $0\notin\sigma_{\rm ess}(\Delta^p_{\tilde{M}})$.

If $p = 0$, we have only p-forms of type I and, since $n > 1$, $0 \notin \sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}})$. By duality, the same result holds for $p = n$.

Finally, let $p = 1$; Propositions 4.11 and 4.12 imply that $\sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}1}) = \emptyset$ whilst

$$\sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}2}) = \left[\left(\frac{n-1}{2}\right)^2 b^2, +\infty\right).$$

Since, in view of Theorem 6.2, 0 cannot be an eigenvalue, following the same argument as before we see that if $0 \in \sigma_{\mathrm{ess}}(\Delta^1_{\tilde{M}})$, then either $0 \in \sigma_{\mathrm{ess}}(\Delta^0_{\tilde{M}1})$ or $0 \in \sigma_{\mathrm{ess}}(\Delta^2_{\tilde{M}2})$. But this is in contradiction with Proposition 4.11 or with Proposition 4.12. Hence, $0 \notin \sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}})$ for any $p \in [0, n]$.

3. Let $a < -1$ and $b > 0$; again, since $N = \mathbb{S}^{n-1}$, the p-th Betti number of N is different from zero if and only if $p = 0$ or $p = n-1$.

Then, if $p \notin \{0, 1, n-1, n\}$, by Propositions 5.4 and 5.5

$$\sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}1}) = \sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}2}) = \emptyset.$$

Since, by Theorem 6.2, 0 cannot be an eigenvalue of $\Delta^p_{\tilde{M}}$, if $0 \in \sigma_{\mathrm{ess}}(\Delta^p_{\tilde{M}})$ there exist a sequence $\{\mu_k\}$ of positive eigenvalues of $\Delta^p_{\tilde{M}3}$ and a corresponding sequence $\{\Phi_k\}$ of normalized, mutually orthogonal eigenforms of $\Delta^p_{\tilde{M}3}$ such that $\mu_k \to 0$ as $k \to +\infty$. Then, by Proposition 6.3, either $0 \in \sigma_{\mathrm{ess}}(\Delta^{p-1}_{\tilde{M}1})$ or $0 \in \sigma_{\mathrm{ess}}(\Delta^{p+1}_{\tilde{M}2})$. Since $1 < p < n-1$, we get a contradiction with Propositions 5.4 and 5.5.

If, on the contrary, $p \in \{0, 1, n-1, n\}$, again by Propositions 5.4 and 5.5, $0 \in \sigma_{\mathrm{ess}}(\Delta_{\tilde{M}})$. □

Since the essential spectra of $\tilde{M}$ and of M coincide, combining the results of section 4 with Proposition 6.4, we finally get:

Theorem 6.5. *Let M be the unitary ball $B(\underline{0}, 1)$ in $\mathbb{R}^n$ endowed with a Riemannian metric which, in a tubular neighbourhood of the boundary $\mathbb{S}^{n-1}$, is given by*

$$d\sigma^2 = e^{-2(a+1)t}\, dt^2 + e^{-2bt}\, d\theta^2_{\mathbb{S}^{n-1}},$$

where $a \leq -1$, $t = settanh(\|x\|)$ and $d\theta^2_{\mathbb{S}^{n-1}}$ is the standard Riemannian metric on $\mathbb{S}^{n-1}$. Then

(1) *if $a = -1$ and $b < 0$, if $p \neq \frac{n}{2}$*

$$\sigma_{\mathrm{ess}}(\Delta^p_M) = \left[\min\left\{\left(\frac{n-2p-1}{2}\right)^2 b^2, \left(\frac{n-2p+1}{2}\right)^2 b^2\right\}, +\infty\right),$$

whilst if $p = \frac{n}{2}$

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = \{0\} \cup \left[\frac{b^2}{4}, +\infty\right);$$

(2) *if* $a = -1$ *and* $b = 0$, *then for every* $p \in [0, n]$

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = [\overline{\lambda}, +\infty),$$

where $\overline{\lambda}$ *is the minimum between the smallest eigenvalue of* $\Delta_{\mathbb{S}^{n-1}}^p$ *and the smallest eigenvalue of* $\Delta_{\mathbb{S}^{n-1}}^{p-1}$;

(3) *if* $a = -1$ *and* $b > 0$, *if* $1 < p < n-1$

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = \emptyset,$$

whilst if $p \in \{0, 1, n-1, n\}$

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = \left[\left(\frac{n-1}{2}\right)^2 b^2, +\infty\right);$$

(4) *if* $a < -1$ *and* $b < 0$, *then, for every* $p \in [0, n]$, $\sigma_{\mathrm{ess}}(\Delta_M^p) = [0, +\infty)$;

(5) *if* $a < -1$ *and* $b = 0$, *for every* $p \in [0, n]$

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = [\overline{\lambda}, +\infty),$$

where $\overline{\lambda}$ *is the minimum between the smallest eigenvalue of* $\Delta_{\mathbb{S}^{n-1}}^p$ *and the smallest eigenvalue of* $\Delta_{\mathbb{S}^{n-1}}^{p-1}$;

(6) *if* $a < -1$ *and* $b > 0$, *if* $1 < p < n-1$ *then*

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = \emptyset,$$

whilst if $p \in \{0, 1, n-1, n\}$

$$\sigma_{\mathrm{ess}}(\Delta_M^p) = [0, +\infty).$$

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